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A MATHEMATICAL MODEL OF ARCHIMEDES' CATTLE PROBLEM

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ABSTRACT:

This paper sheds light on the complex interrelationships between history and mathematical developments by examining Archimedes' Cattle Problem. With modern computing capacities, what seemed like an almost impossible computational problem in Archimedes day is no longer such a seemingly intractable problem. This paper demonstrates the formulation of Archimedes' Cattle Problem through an Excel spreadsheet utilizing an Excel add-in optimizing tool. After providing the solution to Archimedes' Cattle Problem, the paper then argues that intractable mathematical solutions of bygone eras become facilitated over time by the use of new instruments. The paper concludes by noting that in certain cases it is difficult to separate the virtues of the instrument from those of the observer. Had modern computing capacity been available in Archimedes day, the solution to his cattle problem would have been readily at hand.

Key words: *Archimedes, Cattle Problem, Optimization, Math History.*

INTRODUCTION

The Cattle Problem, which is generally attributed to Archimedes, has intrigued mathematicians of modern times since Gotthold Ephraim Lessing discovered a Greek manuscript in 1773. According to some math historians, history can provide a link between understanding a mathematical concept and its application (Swetz , Fauvel, Johansson, Katz, Bekken (1995).

There is perhaps no better example of trying to understand the complex interrelations between history and mathematical developments than Archimedes' Cattle Problem. With the use of modern computing capacities what seemed like an impossible computational problem in Archimedes' time is skillfully handled today with the use of high-powered optimization programs.

In this paper, we demonstrate the formulation of Archimedes' Cattle Problem through an Excel spreadsheet, which utilizes Excel add-in optimizing tools developed by Frontline Systems. Before presenting the mathematical formulation and ultimate solution of Archimedes Cattle Problem, we first examine Archimedes mathematical achievements followed by a brief history of the Cattle Problem. We then present the mathematical model followed by the spreadsheet results. The paper concludes with a brief statement about the importance of understanding mathematical developments through an historical prism.

THE MATHEMATICAL CONTRIBUTIONS OF ARCHIMEDES

Archimedes (287-212 B.C.) is considered, along with Newton and Gauss, to be one of the three greatest mathematicians in history (Bell, 1937; Hart, 1978). Not only did he excel in mathematics, but he also had a formidable intellect in engineering and science. Hirshfeld, one of Archimedes' biographers, credits him with discovering calculus before Newton (Publishers Weekly, 2009). Bell (1986) gives a condensed list of Archimedes' more significant accomplishments:

1. Estimated with great precision the value of pi (π) to be between 3 1/7 and 3 10/71.
2. Discovered the area of a circle to be equal to π times the square of the circle's radius.
3. Developed the mathematics to measure parabolas, spheres and cylinders.
4. Formulated the principle of buoyancy and the law of the lever.
5. Invented the water screw, which is still utilized in modern irrigation.
6. Invented a miniature planetarium, possibly using water for motion, which imitated the movement of earth, moon and the other five known planets: Mercury, Venus, Mars, Jupiter, and Saturn.

Archimedes was a famous mathematician during and after his lifetime. Archimedes lived the last years of his life in Syracuse, which was a city under siege. It is said that his king ordered him to invent effective "futuristic" war machines. Apparently, his efforts met with considerable success in that his machines helped repel the Roman invaders for years (Marchant 2009). Further, it is said that the Roman commander ordered his capture but that the orders were never carried out (Columbia Electronic Encyclopedia, 2009). His fame has increased markedly through the centuries, so much so that he is now considered one of the most famous scientists in the history of humanity (Publishers Weekly, 2009; Hart, 1987). At one point, the European Union offered a 50,000 euro grant for winning the "Archimedes Prize" (Education and Training, 2001; European Commission Community Research, 2002). A photograph of a classic Greek bust of Archimedes is included in the Appendix.

Archimedes' work is often used as source material and inspiration for scholarly articles. The way solid objects behave when floated in liquid is presented as a direct outgrowth of Archimedes' principle of buoyancy (McCuan, 2009). Abu-Saymeh and Hajja (2008) compiled a generalization of Archimedes' arbelos and its contents. Sangwin (2008) discusses the ellipsograph of Archimemdes, which he considers relevant to today's mechanical engineers. Both Shepard (2008) and Burn (2005) have written recently about the ways Archimedes' used, and may have used, mathematics and reasoning skills to arrive at his phenomenal results.

HISTORICAL DEVELOPMENT OF THE CATTLE PROBLEM

In 1769, Gotthold Lessing became the librarian of the Herzog August Library in Wolfenbuttel, Germany (Dorre, 1965). The library held many Greek and Latin manuscripts, which he translated and published. One of Lessing's most fascinating translations was attributed to Archimedes and involved the number of cattle belonging to the sun god. The problem, at the time of the publication of the translation, was over two thousand years old. The translation is shown below; the original Greek is reproduced in the Appendix.

Translation

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

In 1880, Amthor published a partial solution to the problem which is a number with 206,545 integers. The three members of the Hillsboro Mathematical Club continued Amthor's work and published the first 31 and the last 12 digits of the smallest total number of cattle consistent with Archimedes' problem (Bell, 1895). Seventy years later, Williams, German and Zarnke (1965) announced the first complete solution to the problem. The model required almost eight hours of computer time. In 1981, Nelson published a solution printout from a Cray 1 computer, which was one of the first supercomputers. The solution contained 206,545 integers, the same number found by Amthor. On this supercomputer, the calculation required ten minutes. Nelson published five additional solutions. One of the solutions required over one million digits.

MATHEMATICAL FORMULATION OF ARCHIMEDES CATTLE PROBLEM

The model formulation of this problem is shown below:

Minimize: WHITEBULLS+BLACKBULLS+SPOTTEDBULLS+BROWNBULLS
+WHITECOWS+BLACKCOWS+SPOTTEDCOWS+BROWNCOWS

Subject to:

WHITEBULLS –((.8333333333333333*BLACKBULLS)+BROWNBULLS) = 0
BLACKBULLS –((.45*SPOTTEDBULLS)+BROWNBULLS) = 0
SPOTTEDBULLS –((0.30952380952381*WHITEBULLS)+BROWNBULLS) = 0
WHITECOWS - .58333333333333333*(BLACKBULLS+BLACKCOWS) = 0
BLACKCOWS - .45*(SPOTTEDBULLS+SPOTTEDCOWS) = 0
SPOTTEDCOWS - 0.3666666666666667*(BROWNBULLS+BROWNCOWS) = 0
BROWNCOWS - 0.30952380952381*(WHITEBULLS+WHITECOWS) = 0
WHITEBULLS >= 1
BLACKBULLS >= 1
SPOTTEDBULLS >= 1
BROWNBULLS >= 1
WHITECOWS >= 1
BLACKCOWS >= 1
SPOTTEDCOWS >=1
BROWNCOWS >=1
BLACKBULLS = integer
WHITEBULLS = integer
SPOTTEDBULLS = integer
BROWNBULLS = integer
BLACKCOWS = integer
WHITECOWS = integer
SPOTTEDCOWS = integer
BROWNCOWS = integer

$$\sqrt{WHITEBULLS+BLACKBULLS} = integer$$

$$\frac{\sqrt{(8(BROWNBULLS+SPOTTEDBULLS+1)} -1}{2} = integer$$

The first twenty-three constraints defined above are all defined in the first paragraph of the translation. Note: we have used the term "SPOTTED instead of "dappled" and we have used "BROWN" instead of "yellow." The final two constraints are found in the second paragraph of the translation. It is these two final constraints which cause the solution to contain a minimum of 206,545 integers. Such a large number requires a computer more powerful than the desktop computer utilized for this model. Most desktop computers in use today are 32 or 64 bit machines. Desktops computers which are 32 bit machines are capable of representing numbers between -2,147,483,648 and 2,147,483,648 (Willis and Newsome, 2005). Even 64 bit computers are only capable of integers ranging from

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−170,141,183, 460,469,231,731,687,303,715,884,105,728 to 170,141,183,460,469,231,731,687 ,303,715 ,884,105,727 (integer (computer science), n.d.). This integer size limitation makes the final two constraints not feasible for the spreadsheet approach to this problem. Only the constraints described in the first paragraph of the translation (the first twenty-three constraints listed) are included. These constraints do present a formidable problem for 32 bit computers.

SPREADSHEET RESULTS

In this paper, we demonstrate the formulation Archimedes Cattle Problem through an Excel spreadsheet, which utilizes Excel add-in optimizing tools developed by Frontline Systems. The add-in tool Solver, which is part of Excel, is capable of solving small optimization models. However, computing requirements of the model require the utilization of a more powerful solver engine than the one provided with Excel. The engine required for a model this complex is Premium Solver Platform (Version7), also developed by Frontline Systems. The initial spreadsheet formulation is displayed in Table 1. The formula view of the model is presented in Table 2. Premium Solver Platform parameters are displayed in Figure 1. The output sections of the spreadsheet solution are shown in Table 3. Row 3 and cells C19:C23 hold the final solution to the first part of Archimedes Cattle Problem.

Table 1
Archimedes' Cattle Problem Initial Spreadsheet Formulation

1\A	B	C	D	E	F	G	H	I
2	White Bulls	Black Bulls	Spotted Bulls	Brown Bulls	White Cows	Black Cows	Spotted Cows	Brown Cows
3	0	0	0	0	0	0	0	0
4	value	0.833333333333333						
5	black+brown	0	white	0				
6	value	0.45						
7	spotted+brown	0	black	0				
8	value	0.30952380952381000						
9	white+brown	0	spotted	0				
10	value	0.58333333333333						
11	white	0	black	0				
12	value	0.45						
13	black	0	spotted	0				
14	value	0.366666666666667						
15	spotted	0	brown	0				
16	value	0.30952380952381						
17	brown	0	white	0				
18								
19	Total	0						
20	White	0						
21	Black	0						
22	Spotted	0						
23	Brown	0						

Table 2
Archimedes' Cattle Problem Formula View Spreadsheet

1\A2	B	C	D	E	F	G	H	I
2	White Bulls	Black Bulls	Spotted Bulls	Brown Bulls	White Cows	Black Cows	Spotted Cows	Brown Cows
3	0	0	0	0	0	0	0	0
4	value	=1/2+1/3						
5	black+brown	=C4*C3+E3	white	=B3-C5				
6	value	=1/4+1/5						
7	spotted+brown	=C6*D3+E3	black	=C7-C3				
8	value	=1/6+1/7						
9	white+brown	=C8*B3+E3	spotted	=C9-D3				
10	value	=1/3+1/4						
11	white	=C10*(C3+G3)	black	=F3-C11				
12	value	=1/4+1/5						
13	black	=C12*(D3+H3)	spotted	=G3-C13				
14	value	=1/5+1/6						
15	spotted	=C14*(E3+I3)	brown	=H3-C15				
16	value	=1/6+1/7						
17	brown	=C16*(B3+F3)	white	=I3-C17				
18								
19	Total	=SUM(B3:I3)						
20	White	=B3+F3						
21	Black	=C3+G3						
22	Spotted	=D3+H3						
23	Brown	=E3+I3						

Figure 1
Archimedes' Cattle Problem Premium Solver Parameters

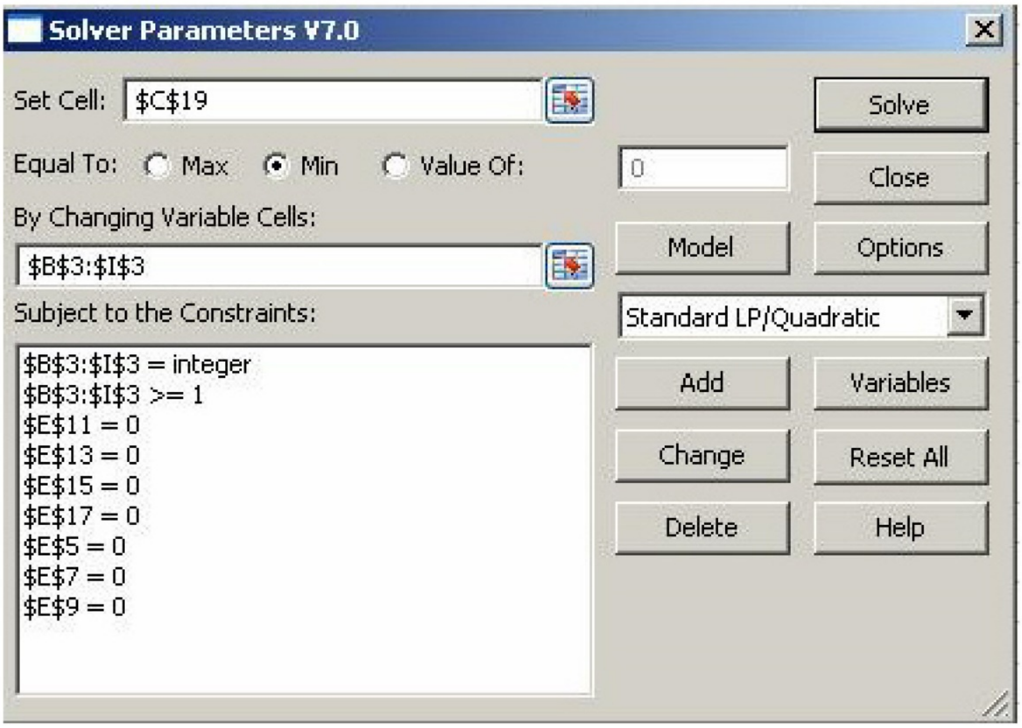

Table 3
Archimedes' Cattle Problem Spreadsheet Solution

1\A	B	C	D	E	F	G	H	I
2	White Bulls	Black Bulls	Spotted Bulls	Brown Bulls	White Cows	Black Cows	Spotted Cows	Brown Cows
3	10,366,482	7,460,514	7,358,060	4,149,387	7,206,360	4,893,246	3,515,820	5,439,213
4	value	0.833333333333333						
5	black+brown	10,366,482	white	0				
6	value	0.45						
7	spotted+brown	7,460,514	black	0				
8	value	0.30952380952381000						
9	white+brown	7,358,060	spotted	0				
10	value	0.583333333333333						
11	white	7,206,360	black	0				
12	value	0.45						
13	black	4,893,246	spotted	0				
14	value	0.366666666666667						
15	spotted	3,515,820	brown	0				
16	value	0.30952380952381						
17	brown	5,439,213	white	0				
18								
19	Total	50,389,082						
20	White	17,572,842						
21	Black	12,353,760						
22	Spotted	10,873,880						
23	Brown	9,588,600						

The final solution to Archimedes' Cattle Problem is summarized in Table 4.

Table 4
Final Solution Archimedes Cattle Problem

White Bulls	10,366,482	Black Bulls	7,460,514	Spotted Bulls	7,358,060	Brown Bulls	4,149,387
White Cows	7,206,360	Black Cows	4,893,246	Spotted Cows	3,515,820	Brown Cows	5,439,213
White Total	17,572,842	Black Total	12,353,760	Spotted Total	10,873,880	Brown Total	9,588,600
Total	50,389,082						

SUMMARY AND CONCLUSION

Among math historians there is a vigorous debate about the extent to which mathematical developments are determined either by outside circumstances or by a kind of internal necessity? According to one historian, the deterministic theory of mathematical progress remains insufficient unless corrected and tempered by historical circumstance (Sarton, 1936). However, Sarton cautions that although "the concatenations of mathematical ideas are not divorced from life, … it is perhaps more possible for a mathematician than for any other man to secrete himself in a tower of ivory." (Sarton, 1936).

The Archimedes' Cattle Problem is a fascinating historical problem in mathematics. The fact that it was formulated by a man over 2,200 years ago is intriguing in itself. It is a good example of an application of optimization that proved to be challenging even today with modern computer technology. It should be an interesting and challenging optimization exercise for researchers in the fields of operations analysis and applied mathematics.

Without conclusively accepting or rejecting either side of the debate regarding the history of mathematical developments, we nonetheless believe that mathematical solutions may become possible or facilitated by the use of new instruments. And, in certain cases it is difficult to separate the virtues of the instrument from those of the observer. Had modern computing capacity been available in Archimedes day, the solution to his cattle problem would have been readily at hand. Yet, at the same time, one does not have to be a student of math history to know that Archimedes was a true genius.

EPILOGUE

Libraries have discovered palimpsests that contained original works by Archimedes but that were later written over with religious works. Hirschfeld (2010) chronicles the discovery and restoration of a palimpsest containing seven of Archimedes' treatises. Findings from the so-called Archimedes Palimpsest have been assembled into a well-reviewed database ("Reference Reviews", 2007) found at http://www.archimedespalimpsest.org/ digitalproduct1 .html.

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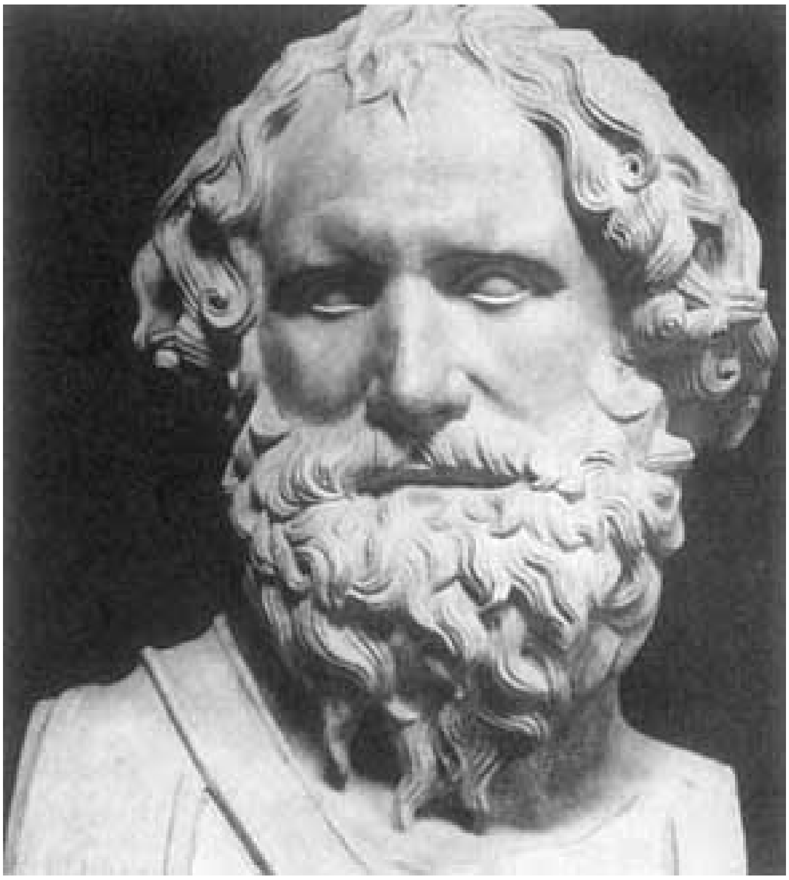
Appendix

Figure 2

Bust of Archimedes



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Greek Statement of Archimedes Cattle Problem

Πρόβλημα,

ὅπερ Ἀρχιμήδης ἐν ἐπιγράμμασιν εὑρὼν τοῖς ἐν Ἀλεξανδρείᾳ περὶ ταῦτα πραγματευομένοις ζητεῖν ἀπέστειλεν ἐν τῇ πρὸς Ἐρατοσθένην τὸν Κυρηναῖον ἐπιστολῇ.

Πληθὺν Ἠελίοιο βοῶν, ὦ ξεῖνε, μέτρησον
φροντίδ᾽ ἐπιστήσας, εἰ μετέχεις σοφίης,
πόσση ἄρ᾽ ἐν πεδίοις Σικελῆς ποτ᾽ ἐβόσκετο νήσου
Θρινακίης τετραχῇ στίφεα δασσαμένη
χροίην ἀλλάσσοντα˙ τὸ μὲν λευκοῖο γάλακτος,
κυανέῳ δ᾽ ἕτερον χρώματι λαμπόμενον,
ἄλλο γε μὲν ξανθόν, τὸ δὲ ποικίλον. ἐν δὲ ἑκάστῳ
στίφει ἔσαν ταῦροι πλήθεσι βριθόμενοι
συμμετρίης τοιῆσδε τετευχότες˙ ἀργότριχας μὲν
κυανέων ταύρων ἡμίσει ἠδὲ τρίτῳ
καὶ ξανθοῖς σύμπασιν ἴσους, ὦ ξεῖνε, νόησον,
αὐτὰρ κυανέους τῷ τετράτῳ τε μέρει
μικτοχρόων καὶ πέμπτῳ, ἔτι ξανθοῖσί τε πᾶσιν.
τοὺς δ᾽ ὑπολειπομένους ποικιλόχρωτας ἄθρει
ἀργεννῶν ταύρων ἕκτῳ μέρει ἑβδομάτῳ τε
καὶ ξανθοῖς αὖτις πᾶσιν ἰσαζομένους.
θηλείαισι δὲ βουσὶ τάδ᾽ ἔπλετο˙ λευκότριχες μὲν
ἦσαν συμπάσης κυανέης ἀγέλης
τῷ τριτάτῳ τε μέρει καὶ τετράτῳ ἀτρεκὲς ἶσαι˙
αὐτὰρ κυάνεαι τῷ τετράτῳ τε πάλιν
μικτοχρόων καὶ πέμπτῳ ὁμοῦ μέρει ἰσάζοντο
σὺν ταύροις˙ πάσης δ᾽ εἰς νομὸν ἐρχομένης
ξανθοτρίχων ἀγέλης πέμπτῳ μέρει ἠδὲ καὶ ἕκτῳ
ποικίλαι ἰσάριθμον πλῆθος ἔχον τετραχῇ.
ξανθαὶ δ᾽ ἠριθμεῦντο μέρους τρίτου ἡμίσει ἶσαι
ἀργεννῆς ἀγέλη ἑβδομάτῳ τε μέρει.
ξεῖνε, σὺ δ᾽ Ἠελίοιο βοῶν πόσαι ἀτρεκὲς εἰπών,
χωρὶς μὲν ταύρων ζατρεφέων ἀριθμόν,
χωρὶς δ᾽ αὖ θήλειαι ὅσαι κατὰ χρῶμα ἕκασται,
οὐκ ἄιδρίς κε λέγοι᾽ οὐδ᾽ ἀριθμῶν ἀδαής,
οὐ μήν πώ γε σοφοῖς ἐναρίθμιος. ἀλλ᾽ ἴθι φράζευ
καὶ τάδε πάντα βοῶν Ἠελίοιο πάθη.
ἀργότριχες ταῦροι μὲν ἐπεὶ μιξαίατο πληθὺν
κυανέοις, ἵσταντ᾽ ἔμπεδον ἰσόμετροι
εἰς βάθος εἰς εὖρός τε, τὰ δ᾽ αὖ περιμήκεα πάντῃ
πίμπλαντο πλίνθου Θρινακίης πεδία.
ξανθοὶ δ᾽ αὖτ᾽ εἰς ἓν καὶ ποικίλοι ἀθροισθέντες
ἵσταντ᾽ ἀμβολάδην ἐξ ἑνος ἀρχόμενοι
σχῆμα τελειοῦντες τὸ τρικράσπεδον οὔτε προσόντων
ἀλλοχρόων ταύρων οὔτ᾽ ἐπιλειπομένων.
ταύτα συνεξευρὼν καὶ ἐνὶ πραπίδεσσιν ἀθροίσας
καὶ πληθέων ἀποδοὺς, ὦ ξένε, πὰντα μέτρα
ἔρχεο κυδιόων νικηφόρος, ἴσθι τε πάντως
κεκριμένος ταύτῃ ὄμπνιος ἐν σοφὶῃ.