BILEVEL PROGRAMMING PROBLEM WITH FUZZY PARAMETERS: A FUZZY GOAL PROGRAMING APPROACH

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Abstract:

This paper describes how fuzzy goal programming can be efficiently used to solve bilevel programming problems with fuzzy parameters. In the model formulation of the problem, the tolerance membership functions for the fuzzily described objective functions of decision makers are defined by determining individual optimal solution of each of the level decision makers. Since the objectives are potentially conflicting in nature, a possible relaxation of the upper level and lower level decision are considered by providing preference bounds on the decision variables for avoiding decision deadlock. Then fuzzy goal programming approach is used for achieving highest degree of each of the membership goals by minimizing negative deviational variables. Three fuzzy goal programming models are presented. Distance function is used to identify which fuzzy goal programming models offers better optimal solution. A numerical example is presented to demonstrate the potential use of the proposed approach.

Keywords: bilevel programming problem, fuzzy goal programming, fuzzy parameters, deviational variables, distance functions

1. Introduction

Decision making within a hierarchical organization may be characterized by an attempt to satisfy a set of potentially conflicting objectives of different decision making units situated in hierarchical levels as completely as possible in an environment comprised of a set of finite resources, conflicting interest and a set of constraints in order to deal with the situation in which all objectives can not be completely and simultaneously satisfied. Constraints and objectives may be fuzzily described. Hierarchical optimization or multilevel programming (MLP) techniques are extensions of Stackleberg games for solving decentralized planning problems with multiple decision makers (DMs) in a hierarchical organization. Bilevel programming problem (BLPP) is a special case of Multilevel programming problems (MLPPs) of a large hierarchical decision system. Bilevel organization has following common characteristics: two decision makers namely, upper level decision maker (ULDM) and lower level decision maker (LLDM) are located at two different levels; the execution of decision is sequential, from upper level to lower level; each DM independently controls only a set of decision variables and is interested in optimizing his or her objective functions. Although ULDM independently optimizes his or her own objective functions, the decision may be affected by the reaction of the LLDM. Therefore, decision dead lock arises frequently in the decision making situation.

The formal formulation of the linear BLPP is defined by Candler and Towmsly [9] as well as Fortuny-Amat and McCarl [10]. During the last three decades, BLPP as well as MLPP in general for hierarchical decentralized planning problems have been deeply studied in [1-5, 7-11, 13, 17-22] and many methodologies have been proposed to solve them potentially such as economic systems, government policy, warfare, etc. The classical approaches developed so far, for BLPPs have been surveyed by Wen and Tsu [22]. Most of these methods are based on vertex enumeration [8] and transformation approach [9]. The former is to seek a compromise vertex by simplex algorithm based on adjusting upper level control variables. It is rather inefficient, especially for large size problems. The latter involves transforming the lower level programming problem to be constraints of the upper level by its Kuhn-Tucker (K-T) conditions or penalty function. Due to the presence of non-linear or Lagrangian terms appearing in the constraints, the auxiliary problems become complex and sometimes unmanageable. These methods are suitable for crisp environment. Sakawa et al. [17] presented an interactive fuzzy mathematical programming for linear MLPP with fuzzy parameters. Lai [11] at first developed an effective fuzzy approach by using the concept of tolerance membership functions for solving MLPPs in 1996. Shih et al. [18] extended Lai's concept by using non-compensatory max-min aggregation operator for solving MLPPs. Shih and Lee [19] further extended Lai's concept by introducing the compensatory fuzzy operator for solving MLPPs. Sinha [20] studied alternative MLP technique based on fuzzy mathematical programming (FMP). The basic concept of these fuzzy approaches is almost same and re-evaluation of the problem repeatedly by redefining the elicited membership values is essentially needed in the solution search process to obtain a satisfactory solution. So, computational load is also inherently involved in the fuzzy approaches developed so far. Pramanik and Roy [15] proposed fuzzy goal programming (FGP) approach to MLPPs using deviational variables.

Pramanik and Roy [16] proposed FGP approach for solving multi-objective transportation problem with fuzzy parameters. Pramanik and Roy [14] also developed priority based FGP approach for solving multi-objective transportation problem with fuzzy parameters. In this paper, FGP due to Pramanik and Roy [14, 16] is slightly modified and applied for solving BLPP with fuzzy parameters. Three FGP models are formulated for solving BLPPs. Distance function is used for identifying which FGP model offers better optimal solution.

2. Preliminaries

Some basic definitions are here given that will be used in the paper.

Definition 2.1 A fuzzy set $\tilde{A}$ in $\overline{X}$ is defined by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in \overline{X}\}$, where $\mu_{\tilde{A}}(x): \overline{X} \to [0, 1]$ is called the membership function of $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is the degree of membership to which $x \in \tilde{A}$.

Definition 2.2 Union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ with respective membership functions $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{B}}(x)$ is defined by a fuzzy set $\tilde{C}$ whose membership function is defined by $\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{C}}(x) = \max[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]$, $x \in \overline{X}$.

Definition 2.3 Intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ with respective membership functions $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{B}}(x)$ is defined by a fuzzy set $\tilde{C}$ whose membership function is defined by $\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{C}}(x) = \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]$, $x \in \overline{X}$.

Definition 2.4 The α-cut of a fuzzy set $\tilde{A}$ of $\overline{X}$ is a non-fuzzy set denoted by ${}^{\alpha}A$ is defined by a subset of all elements $x \in \overline{X}$ such that their membership functions exceed or equal to a real number $\alpha \in [0, 1)$, i.e. ${}^{\alpha}A = \left[x : \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1), \forall x \in \overline{X}\right]$.

3. Formulation of fuzzy goal programming having fuzzy parameters

Consider the following fuzzy optimization problem:

Minimize $\tilde{Z}(\overline{X}) = \left(\tilde{C}_1 \overline{X}, \tilde{C}_2 \overline{X}, ..., \tilde{C}_K \overline{X}\right)^T$ (1)

subject to $\overline{X} \in S = \{\overline{X} \in \Re^n \mid \tilde{A}\overline{X} * \tilde{B}, \overline{X} \geq \overline{0}\}$, (2)

where $\tilde{C}_k (k = 1, 2, ..., K)$ are n-dimensional vector, $\tilde{B}$ is an m-dimensional vector, $\tilde{A}$ is an $m \times n$ matrix, and $\tilde{C}_k$, $\tilde{B}$, and $\tilde{A}$ are fuzzy numbers. Here, the symbol $*$ denotes respectively $\geq$, $=$, and $\leq$. $\overline{X} = (X_1, X_2, ..., X_n)^T$. Consider that the problem represented by (1) has fuzzy coefficients, which have possibilistic distributions. Assume that ${}^{\alpha}\overline{X}$ be a solution of (1) where $\alpha \in [0,1)$ represents the level of possibility at which all fuzzy coefficients is feasible.

Let ${}^{\alpha}(\tilde{R})$ be the α-cut of a fuzzy number $\tilde{R}$ defined by

$${}^{\alpha}(\tilde{R}) = \left\{r \in \mathrm{Supp}(\tilde{R}) \| \mu_{\tilde{R}}(r) \geq \alpha, \alpha \in [0,1)\right\} \quad (3)$$

where $\mathrm{Supp}(\tilde{R})$ is the support of $\tilde{R}$. Let ${}^{\alpha}(\tilde{R})^L$ and ${}^{\alpha}(\tilde{R})^U$ be the lower bound and upper bound of the α-cut of $\tilde{R}$ respectively such that ${}^{\alpha}(\tilde{R})^L \leq {}^{\alpha}(\tilde{R}) \leq {}^{\alpha}(\tilde{R})^U$ (4)

Then, for a prescribed value of α, for minimization-type objective function, $\tilde{Z}_k(\overline{X})$ (k = 1, 2, ..., K) can be replaced by the lower bound of its α-cut i.e.

$$^{\alpha}\left(\tilde{Z}_k(\overline{X})\right)^L = \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^L X_j \qquad (5)$$

Similarly, for maximization-type objective function, $\tilde{Z}_k(\overline{X})$ (k = 1, 2, ..., K) can be replaced by the upper bound of its α-cut i.e. $^{\alpha}\left(\tilde{Z}_k(\overline{X})\right)^U = \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^U X_j$ (6)

For inequality constraints

$$\sum_{j=1}^{n} \tilde{A}_{ij} X_j \geq \tilde{B}_i \;, i = 1, 2, \ldots, m_1, \qquad (7)$$

and $\sum_{j=1}^{n} \tilde{A}_{ij} X_j \leq \tilde{B}_i$, $i = m_1+1, \ldots, m_2$, (8)

can be rewritten by the following constraints:

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^U X_j \geq {}^{\alpha}\left(\tilde{B}_i\right)^L, i = 1, 2, \ldots, m_1 \qquad (9)$$

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^L X_j \leq {}^{\alpha}\left(\tilde{B}_i\right)^U, i = m_1+1, \ldots, m_2 \qquad (10)$$

For fuzzy equality constraints

$$\sum_{j=1}^{n} \tilde{A}_{ij} X_j = \tilde{B}_i \quad i = m_2+1, \ldots, m, \qquad (11)$$

can be replaced by two equivalent constraints

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^L X_j \leq {}^{\alpha}\left(\tilde{B}_i\right)^U \qquad (12)$$

and $\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^U X_j \geq {}^{\alpha}\left(\tilde{B}_i\right)^L$ (13)

For proof of equivalency of (11) with (12) and (13), see Lee and Li [12].

Therefore, for a prescribed value of α, the problem represented by (1) can be transformed to the following problem:

Minimize $^{\alpha}\left(\tilde{Z}_k(\overline{X})\right)^L = \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^L X_j$, k = 1, 2, ..., K, (14)

subject to $\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^U X_j \geq {}^{\alpha}\left(\tilde{B}_i\right)^L$, $i = 1, 2, \ldots, m_1, m_2+1, \ldots, m$, (15)

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^L X_j \leq {}^{\alpha}\left(\tilde{B}_i\right)^U, i = m_1+1, \ldots, m_2, m_2+1, \ldots, m, \qquad (16)$$

$X_j \geq 0$, j = 1, 2, ..., n. (17)

For simplicity, denote the system constraints (15), (16) and (17) as S`.

For a prescribed value of α, the problem (14) reduces to a deterministic linear programming problem with multiple objectives, which can be solved by applying the FGP proposed By Pramanik and Roy [14, 16].

The resulting membership functions for minimization-type objective functions are defined as:

$$\mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right) = \left[{}^{\alpha}\left(Z_k\right)^{-} - \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^{L} X_j\right] \Big/ \left[{}^{\alpha}\left(Z_k\right)^{-} - {}^{\alpha}\left(Z_k\right)^{O}\right], \ k = 1, 2, \ldots, K, \quad (18)$$

where the aspired level ${}^{\alpha}\left(Z_k\right)^{O}$ and highest acceptable level ${}^{\alpha}\left(Z_k\right)^{-}$ are ideal and anti-ideal solutions, respectively, which can be obtained by solving each of the following problem independently:

$${}^{\alpha}\left(Z_k\right)^{O} = \underset{X\in S`}{\text{Minimize}} \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^{L} X_j, \quad (19)$$

$${}^{\alpha}\left(Z_k\right)^{-} = \underset{X\in S`}{\text{Maximize}} \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^{U} X_j, \ k = 1, 2, \ldots, K. \quad (20)$$

For maximization-type objective function, ideal and anti-ideal solutions can be similarly obtained.

Assume that all of the fuzzy coefficients are trapezoidal fuzzy numbers. Trapezoidal fuzzy number $\tilde{R}$ can be defined as:

$\tilde{R} = (r^{(1)}, r^{(2)}, r^{(3)}, r^{(4)})$ and the membership function of the trapezoidal fuzzy number (see Figure1) will be interpreted as follows:

$$\mu_{\tilde{R}}(r) = \left\{\begin{array}{ll} 0, & r \le r^{(1)}, \\ \dfrac{r - r^{(1)}}{r^{(2)} - r^{(1)}}, & r^{(1)} \le r \le r^{(2)}, \\ 1, & r^{(2)} \le r \le r^{(3)}, \\ \dfrac{r^{(4)} - r}{r^{(4)} - r^{(3)}}, & r^{(3)} \le r \le r^{(4)}, \\ 0, & r \ge r^{(4)} \end{array}\right\} \quad (21)$$

So, an α-cut of $\tilde{R}$ [12] can be expressed by the following interval

$${}^{\alpha}\left(\tilde{R}\right) = \left[{}^{\alpha}\left(\tilde{R}\right)^{L}, {}^{\alpha}\left(\tilde{R}\right)^{U}\right] = \left[(r^{(2)} - r^{(1)})\alpha + r^{(1)}, -(r^{(4)} - r^{(3)})\alpha + r^{(4)}\right] \quad (22)$$

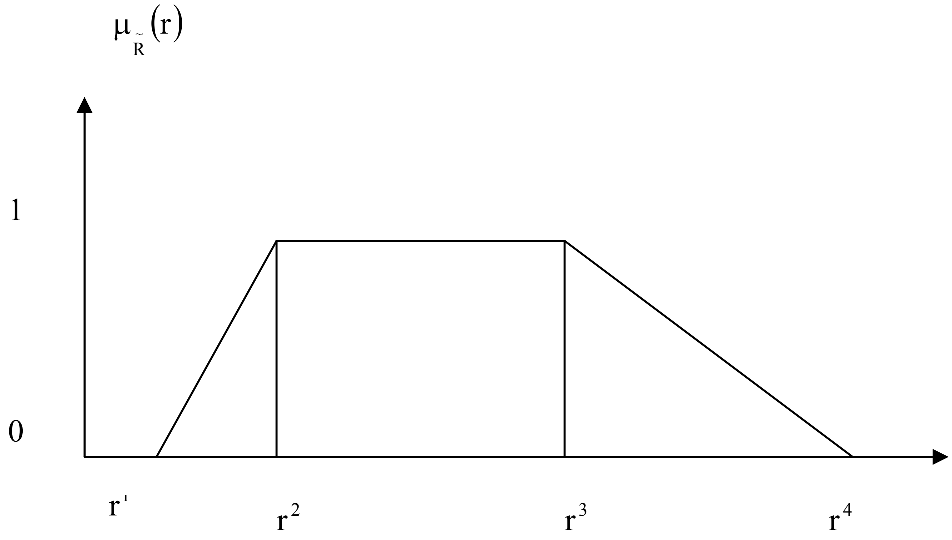

Figure 1. Trapezoidal fuzzy number $\tilde{R}$ = ($r^{(1)}$, $r^{(2)}$, $r^{(3)}$, $r^{(4)}$)

It is to be noted that when $r^{(2)} = r^{(3)}$, $\tilde{R}$ transforms into the triangular fuzzy number, specified by ($r^{(1)}$, $r^{(2)} = r^{(3)}$, $r^{(4)}$);

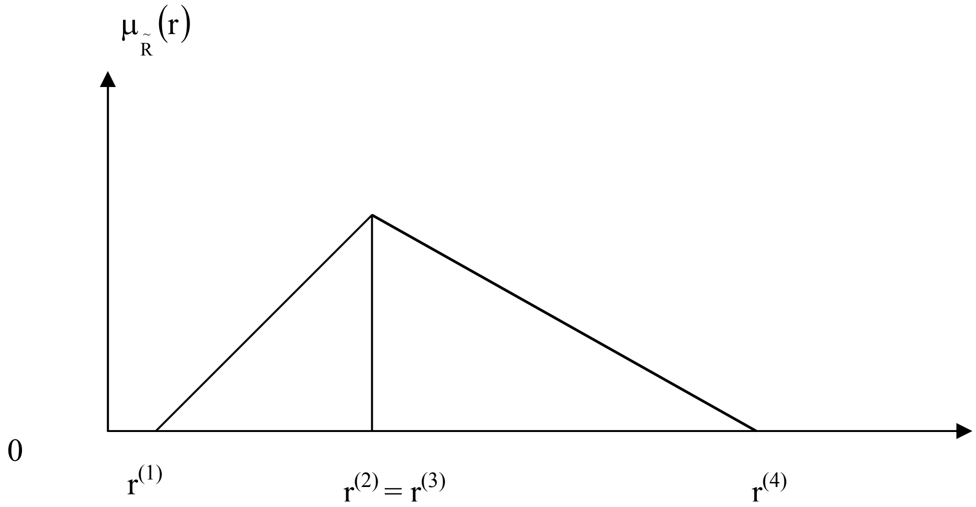

Figure 2. Triangular fuzzy number $\tilde{R}$ = ($r^{(1)}$, $r^{(2)}$= $r^{(3)}$, $r^{(4)}$)

For given value of α, the main interest of the decision maker is to maximize the degree of membership function of the objectives and constraints to the respective fuzzy goals i.e.

Maximize $\mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right)$ (23)

subject to $0 \le \mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right) \le 1$, (24)

$\overline{X} \in S`$ (25)

Here one can adopt Bellman- Zadeh's [6] fuzzy decision based on minimum operator.

$$\mu_D(\overline{X}) = \mathop{\wedge}_{k=1}^{K} \mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right),\ \overline{X} \in S` \tag{26}$$

The problem (23) can be transformed to the following problem:

Max γ (27)

$$\gamma \le \mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right),\ k = 1, 2, \dots, K, \tag{28}$$

$$\overline{X} \in S` \tag{29}$$

$$\mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right) \in [0, 1] \tag{30}$$

where γ represents minimal acceptable degree of objectives.

The problem (27) can be transformed into linear goal program. The highest value of a membership function is 1. So for the defined membership functions in (27), the flexible membership goals having the aspired level unity can be represented as:

$$\mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)\right) + D_k^- - D_k^+ = 1,\ k = 1, 2, \dots, K. \tag{31}$$

Here D_k^-, D_k^+ are negative and positive deviational variables with $D_k^- \times D_k^+ = 0$. (32)

Any positive deviation from a fuzzy goal implies the full achievement of the membership value unity. Therefore, we assign only negative deviational variables in the achievement function. Therefore, (31) can be written as

$$^{\alpha}\mu_k\left(Z_k\left(\overline{X}\right)\right) + D_k^- = 1 \tag{32}$$

Pramanik and Roy [15] used inequality sign for FGP model that is $^{\alpha}\mu_k\left(Z_k\left(\overline{X}\right)\right) + D_k^- \ge 1$ for dealing with multilevel programming problem. Since $D_k^- \ge 0$ and there is no possibility of positive deviation, $D_k^- = 0$. Therefore, we omit the extra positive deviational variable and use equality sign as (32).

Under the framework of minsum goal programming, the FGP model of the problem can be explicitly formulated as:

Model (1):

Minimize λ (33)

subject to

$$\frac{\left[{}^{\alpha}(Z_k)^- - \sum_{j=1}^{n} {}^{\alpha}\left(\tilde{C}_{kj}\right)^L X_j\right]}{\left[{}^{\alpha}(Z_k)^- - {}^{\alpha}(Z_k)^O\right]} + D_k^- = 1,\ k = 1, 2, \dots, K, \tag{34}$$

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^U X_j \ge {}^{\alpha}\left(\tilde{B}_i\right)^L,\ i = 1, 2, \dots, m_1, m_2+1, \dots, m, \tag{35}$$

$$\sum_{j=1}^{n} {}^{\alpha}\left(\tilde{A}_{ij}\right)^L X_j \le {}^{\alpha}\left(\tilde{B}_i\right)^U,\ i = m_1+1, \dots, m_2, m_2+1, \dots, m, \tag{36}$$

$$\lambda \ge D_k^-,\ k = 1, 2, \dots, K, \tag{37}$$

$$X_j \ge 0,\ j = 1, 2, \dots, n, \tag{38}$$

$D_k^- \geq 0$, k = 1, 2, ..., K. (39)

Model (IIa): Minimize $\zeta = (\sum_{k=1}^{K} w_k^- D_k^-)$ (40)

and Model (IIb): Minimize $\xi = \sum_{k=1}^{K} D_k^-$ (41)

subject to the constraints given by (34), (35), (36), (38), and (39)

Using the interval expression (22), the problem (33) can be written as:

Minimize λ (42)

subject to

$$\left\{ {}^{\alpha}(Z_k)^- - \sum_{j=1}^{n} \left[C_{kj}^{(1)} + \left(C_{kj}^{(2)} - C_{kj}^{(1)} \right) \alpha \right] X_j \right\} \Big/ \left[{}^{\alpha}(Z_k)^- - {}^{\alpha}(Z_k)^O \right] + D_k^- = 1, \; k = 1, 2, \ldots, K, \quad (43)$$

$$\left[A_{ij}^{(4)} - \left(A_{ij}^{(4)} - A_{ij}^{(3)} \right) \alpha \right] X_j \geq B_i^{(1)} + \left(B_i^{(2)} - B_i^{(1)} \right) \alpha, i = 1, \ldots, m_1, m_2 + 1, \ldots, m, \quad (44)$$

$$\left[A_{ij}^{(1)} + \left(A_{ij}^{(2)} - A_{ij}^{(1)} \right) \alpha \right] X_j \leq B_i^{(4)} - \left(B_i^{(4)} - B_i^{(3)} \right) \alpha, \quad i = m_1 + 1, 2, \ldots, m_2, m_2 + 1, \ldots, m, \quad (45)$$

$\lambda \geq D_k^-$, k = 1, 2, ..., K, (46)

$X_j \geq 0$, j = 1, 2, ..., n, (47)

$D_k^- \geq 0$, k = 1, 2, ..., K. (48)

Similarly, using the interval expression (22), the problem (40) and (41) can be written as

Minimize $\zeta = (\sum_{k=1}^{K} W_k^- D_k^-)$ (49)

and Minimize $\xi = \sum_{k=1}^{K} D_k^-$ (50)

subject to the constraints given by (43), (44), (45), (47), and (48).

Numerical weight W_k^- associated with negative deviational variable is defined by

$W_k^- = 1/\left[{}^{\alpha}(Z_k)^- - {}^{\alpha}(Z_k)^O \right]$, k = 1, 2, ..., K (51)

In Model (IIa), numerical weight W_k^- (k = 1, 2, ..., K)is the reciprocal of the admissible violation constant. The numerical weight associated with negative deviational variable represents the relative importance of achieving the aspired level of the fuzzy goal. The larger admissible violation of constants $\left[{}^{\alpha}(Z_k)^- - {}^{\alpha}(Z_k)^O \right]$ indicates less important k-th fuzzy goal. i.e. the larger numerical weight $W_k^- = 1/\left[{}^{\alpha}(Z_k)^- - {}^{\alpha}(Z_k)^O \right]$, (k = 1, 2, ..., K) indicates the more important of the k-th fuzzy goal. When the numerical weights associated with the negative deviational variables are all equal to unity, then Model (IIa) and Model (IIb) become identical. Therefore, Model (IIb) is a special case of Model (IIa).

4. Formulation of BLPP

A BLPP can be defined as a two- person game with perfect information in which each DM moves sequentially from upper level to lower level. This problem has a nested hierarchical structure with two levels of DMs. We consider a BLPP having maximizing type objective function at each level. Mathematically, the problem can be stated as:

$$\underset{\overline{X}_1}{\text{Maximize}}\, Z_1\left(\overline{X}\right) = \tilde{C}_{11}\,\overline{X}_1 + \tilde{C}_{12}\,\overline{X}_2 \quad \text{(Upper level DM's problem)} \quad (52)$$

$$\underset{\overline{X}_2}{\text{Maximize}}\, Z_2\left(\overline{X}\right) = \tilde{C}_{21}\,\overline{X}_1 + \tilde{C}_{22}\,\overline{X}_2 \quad \text{(Lower level DM's problem)} \quad (53)$$

subject to

$$\tilde{A}_1\,\overline{X}_1 + \tilde{A}_2\,\overline{X}_2 \le \tilde{B} \quad (54)$$

$$\overline{X}_1 \ge \overline{0}, \overline{X}_2 \ge \overline{0}, \quad (55)$$

$\overline{X}_1 = \{X_1^1, X_1^2, \ldots, X_1^{N_1}\}^T$: decision variables under the control of ULDM

$\overline{X}_2 = \{X_2^1, X_2^2, \ldots, X_2^{N_2}\}^T$: decision variables under the control LLDM

$\overline{Z} = (Z_1, Z_2)^T$, and T denotes transposition; $\tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix}$ is the vector of coefficient vectors represented by fuzzy parameters.

Where $\tilde{A}_1$ is $M \times N_1$ and $\tilde{A}_2$ is $M \times N_2$ matrix, $\overline{B}$ is the M component column vector. $\overline{X} = \overline{X}_1 \cup \overline{X}_2$ is the set of decision vector, $N = N_1 + N_2$, total number of decision variables in the system and M is the total number of the constraints of the problem. Z_1, Z_2 are linear and bounded.

4.1. Characterization of Membership Function of BLPP

In the decision making situation, each DM is interested in maximizing his or her own objective function. So, the optimal solution of each DM when calculated in isolation would be considered as the aspiration level of each of the respective fuzzy objective goals. For a prescribed value of α, to construct membership function for maximization-type objective function, $\tilde{Z}_i\left(\overline{X}\right)$ (i = 1, 2) can be replaced by the upper bound of its α-cut i.e.

$$^{\alpha}\left(\tilde{Z}_i\left(\overline{X}\right)\right)^U = \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{ij}\right)^U X_j \quad , i = 1, 2 \quad (56)$$

For inequality constraints, $\tilde{A}_1\,\overline{X}_1 + \tilde{A}_2\,\overline{X}_2 \le \tilde{B}$, we write

$$^{\alpha}\left(\tilde{A}_1\right)^L \overline{X}_1 + {}^{\alpha}\left(\tilde{A}_2\right)^L \overline{X}_2 \le {}^{\alpha}\left(\tilde{B}\right)^U \quad (57)$$

Therefore, for a prescribed value of α, the problem reduces to the following problem:

$$\underset{\overline{X}_1}{\text{Maximize}}\ {}^{\alpha}\left(\tilde{Z}_1(\overline{X})\right)^{U} = \underset{\overline{X}_1}{\text{Maximize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{1j}\right)^{U} \overline{X}_j, \quad (58)$$

$$\underset{\overline{X}_2}{\text{Maximize}}\ {}^{\alpha}\left(\tilde{Z}_2(\overline{X})\right)^{U} = \underset{\overline{X}_2}{\text{Maximize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{2j}\right)^{U} \overline{X}_j, \quad (59)$$

subject to

$${}^{\alpha}\left(\tilde{A}_1\right)^{L} \overline{X}_1 + {}^{\alpha}\left(\tilde{A}_2\right)^{L} \overline{X}_2 \le {}^{\alpha}\left(\tilde{B}\right)^{U} \quad (60)$$

$$\overline{X}_1 \ge \overline{0}, \overline{X}_2 \ge \overline{0}, \quad (61)$$

For simplicity, denote the system constraints (60) and (61) as S.

For a prescribed value of α, the fuzzy BLPP reduces to deterministic BLPP, which can be solved by using FGP models discussed in section 3.

Let ($\overline{X}_1^1, \overline{X}_2^1;\ {}^{\alpha}\left(Z_1^B\right)^U$) and ($\overline{X}_1^2, \overline{X}_2^2;\ {}^{\alpha}\left(Z_2^B\right)^U$) be the individual optimal decision of the DMU and DML respectively when calculated in isolation,

where $${}^{\alpha}\left(Z_1^B\right)^U = \underset{\overline{X}\in S}{\text{Maximize}}\ {}^{\alpha}\left(\tilde{Z}_1(\overline{X})\right)^{U} = \underset{\overline{X}\in S}{\text{Maximize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{1j}\right)^{U} \overline{X}_j \quad (62)$$

and $${}^{\alpha}\left(Z_2^B\right)^U = \underset{\overline{X}\in S}{\text{Maximize}}\ {}^{\alpha}\left(\tilde{Z}_2(\overline{X})\right)^{U} = \underset{\overline{X}\in S}{\text{Maximize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{2j}\right)^{U} \overline{X}_j \quad (63)$$

If the individual optimal solutions are identical, then optimal compromise solution is automatically reached. However, this rarely happens due to the conflicting objectives. Then the fuzzy objective goals of the ULDM and LLDM appear as:

$${}^{\alpha}\left(\tilde{Z}_1(\overline{X})\right)^{U} \underset{\sim}{\ge} {}^{\alpha}\left(Z_1^B\right)^U, \quad {}^{\alpha}\left(\tilde{Z}_2(\overline{X})\right)^{U} \underset{\sim}{\ge} {}^{\alpha}\left(Z_2^B\right)^U;$$

To formulate membership functions for the maximization type objective functions, we define:

$${}^{\alpha}\left(Z_1^W\right)^L = \underset{\overline{X}\in S}{\text{Minimize}}\ {}^{\alpha}\left(\tilde{Z}_1(\overline{X})\right)^{L} = \underset{\overline{X}\in S}{\text{Minimize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{1j}\right)^{L} \overline{X}_j \quad (64)$$

$${}^{\alpha}\left(Z_2^W\right)^L = \underset{\overline{X}\in S}{\text{Minimize}}\ {}^{\alpha}\left(\tilde{Z}_2(\overline{X})\right)^{L} = \underset{\overline{X}\in S}{\text{Minimize}} \sum_{j=1}^{2} {}^{\alpha}\left(\tilde{C}_{2j}\right)^{L} \overline{X}_j \quad (65)$$

where ${}^{\alpha}\left(Z_i^B\right)^U$, ${}^{\alpha}\left(Z_i^W\right)^L$, (i = 1, 2) are best and worst or ideal and anti-ideal solutions respectively.

Then the resulting membership functions can be defined as:

$$\mu_i\ {}^{\alpha}\left(Z_i(\overline{X})^U\right) = \begin{cases} 1, & \text{if } {}^{\alpha}\left(Z_i(\overline{X})\right)^U \ge {}^{\alpha}\left(Z_i^B\right)^U \\ \dfrac{{}^{\alpha}\left(Z_i(\overline{X})\right)^U - {}^{\alpha}\left(Z_i^W\right)^L}{{}^{\alpha}\left(Z_i^B\right)^U - {}^{\alpha}\left(Z_i^W\right)^L}, & \text{if } {}^{\alpha}\left(Z_i^W\right)^L \le {}^{\alpha}\left(Z_i(\overline{X})\right)^U \le {}^{\alpha}\left(Z_i^B\right)^U \\ 0, & \text{if } {}^{\alpha}\left(Z_i(\overline{X})\right)^U \le {}^{\alpha}\left(Z_i^W\right)^L \end{cases}, \text{(i= 1, 2)} \quad (66)$$

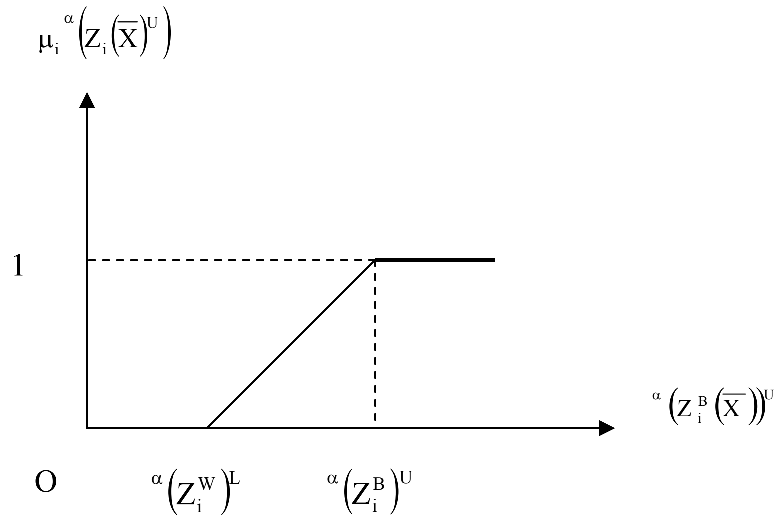

Figure3. Membership function for objective function ${}^{\alpha}\left(\tilde{Z}_i\left(\overline{X}\right)\right)^U$ (i = 1, 2).

In BLPP, Mishra [13] considered arbitrary relaxations on decision variables provided by DMs by providing preference bounds on the decision variables. In the proposed approach, DMs provide the upper and lower bounds on the decision variables under their control. Suppose $\left(\overline{X}_i^B - \overline{R}_i^-\right)$, $\left(\overline{X}_i^B + \overline{R}_i^+\right)$ (i = 1, 2) are the upper and lower bounds of decision vector provided by the i-th level DM where $\overline{X}_i^B$ is the individual best solution when calculated in isolation. Here, $\overline{R}_i^-$ and $\overline{R}_i^+$ are the negative and positive tolerance vectors, which are not necessarily same. Generally $\overline{X}_i$ lies between $\left(\overline{X}_i^B - \overline{R}_i^-\right)$ and $\left(\overline{X}_i^B + \overline{R}_i^+\right)$. DMs may prefer to shift the range of $\overline{X}_i^B$ which may be left of $\overline{X}_i^B$ or right of $\overline{X}_i^B$ only depending on the needs and desires of the level DMs in the decision making situation. For example, if $\overline{X}_i = \overline{0}$, then $\overline{X}_i$ should lie on the right of $\overline{0}$. Then DM should assign $\overline{R}_i^- \le \overline{0}$, $\overline{R}_i^+ \ge \overline{0}$ and $|\overline{R}_i^-| \le |\overline{R}_i^+|$.If the DM wants the shift towards left of $\overline{X}_i^B$, then $\overline{R}_i^-$ should be assigned positive value while $\overline{R}_i^+$ should be assigned a negative value i.e. $\overline{R}_i^- \ge \overline{0}$, $\overline{R}_i^+ \le \overline{0}$ and $|\overline{R}_i^-| \ge |\overline{R}_i^+|$. Similarly, if the shift is required to right of $\overline{X}_i^B$, then DM should assign $\overline{R}_i^- \le \overline{0}$, $\overline{R}_i^+ \ge \overline{0}$ and $|\overline{R}_i^-| \le |\overline{R}_i^+|$.

Therefore, $\left(\overline{X}_i^B - \overline{R}_i^-\right) \le \overline{Xi} \le \left(\overline{X}_i^B + \overline{R}_i^+\right)$, (i=1, 2) (67)

4.2. Formulation of FGP Model

The proposed FGP formulation can be presented as:

Model (I):

Minimize λ (68)

subject to

$$\mu_i^{\alpha}\left(Z_i\left(\overline{X}\right)^U\right) + D_i^- = 1, \text{ (i = 1, 2)} \quad (69)$$

$$\left(\overline{X}_i^B - \overline{R}_i^-\right) \le \overline{Xi} \le \left(\overline{X}_i^B + \overline{R}_i^+\right), \text{ (i =1, 2)} \quad (70)$$

$$^{\alpha}\left(\tilde{A}_1\right)^L \overline{X}_1 + {}^{\alpha}\left(\tilde{A}_2\right)^L \overline{X}_2 \le {}^{\alpha}\left(\tilde{B}\right)^U, \quad (71)$$

$$\lambda \ge D_i^-, \text{ (i = 1, 2)} \quad (72)$$

$$D_i^- \ge 0, \overline{X}_1 \ge \overline{0}, \overline{X}_2 \ge \overline{0} \quad (73)$$

Model (IIa): Minimize $\zeta = \sum_{i=1}^{2} W_i^- D_i^-$ (74)

and Model (IIb): Minimize $\xi = \sum_{i=1}^{2} D_i^-$ (75)

subject to the constraints (69), (70), (71), (73)

The numerical weight $W_i^- = 1/[{}^{\alpha}\left(Z_i^B\right)^U - {}^{\alpha}\left(Z_i^W\right)^L]$ associated with negative deviational variable is determined as discussed in section 3. By solving FGP formulations (68), if the DMs are satisfied with this solution, then a satisficing solution is reached. Otherwise, the DMs should provide new tolerance limits for the control variables until a satisficing solution is reached. In general, considering a set of possible relaxation offered by DMs, the solution becomes satisficing for both level DMs. Similarly, other two FGP formulations are solved.

5. Selection of compromise solution

The concept of utopia point (the ideal-point) and the use the distance function for group decision analysis has been studied by Yu [23]. Since the aspired level of each of the membership goals is unity, the point consisting of the highest membership value of each of the goals would represent the ideal point. The distance function can be defined as:

$$L_P = \left[\sum_{k=1}^{K}\left[1 - \mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)^U\right)\right]^p\right]^{1/p}, \text{ p} \ge 1; \text{ (k = 1, 2, ..., K)}. \quad (76)$$

Where $\mu_k^{\alpha}\left(Z_k\left(\overline{X}\right)^U\right)$ is the membership value for the solution $\overline{X}$. Here, we consider p =1, 2, ∞ only. Now, it can be easily realized that the solution for which L_p is minimum would be the most satisfactory solution. Here, distance function is used only to identify which FGP model (Model I, Model IIa, Model IIb) gives better optimal solution.

6. FGP algorithm for BLPP with fuzzy parameters

Step1. For specified value of α, to construct membership functions for the objective functions of the DMs, the upper and lower bounds of their α-cuts are defined. Similarly, for inequality constraints, upper and lower bounds of their α-cuts are defined.

Step2. Calculate the individual maximum and minimum values for lower and upper α-cuts of the objective functions subject to constraints (60) and (61).

Step3. Determine the weight $W_i^- = 1/[{}^{\alpha}\left(Z_i^B\right)^U - {}^{\alpha}\left(Z_i^W\right)^L]$, (i = 1, 2).

Step4. Construct the membership function $\mu_i^{\alpha}\left(Z_i\left(\overline{X}\right)^U\right)$, (i = 1, 2).

Step5. Consider the preference bounds on the decision vectors provided by the decision makers under their control such that $\left(\overline{X}_i^B - \overline{R}_i^-\right) \le \overline{X}i \le \left(\overline{X}_i^B + \overline{R}_i^+\right)$, (i =1, 2).

Step6. Formulate the three FGP models.

Step7. Solve the three FGP models.

Step8. Compute the distance function for optimal solution obtained from three models.

Step9. Find the optimal solution for which the distance function is minimal. This optimal solution will be the compromise solution for the BLPP.

7. Numerical Example

$$\underset{X_1, X_2}{\text{Maximize}}\, \overline{Z}_1 = \tilde{5}X_1 + \tilde{6}X_2 + \tilde{4}X_3 + \tilde{2}X_4 \qquad \text{(Upper-level)}$$

$$\underset{X_3, X_4}{\text{Maximize}}\, \overline{Z}_2 = \tilde{8}X_1 + \tilde{9}X_2 + \tilde{2}X_3 + \tilde{4}X_4 \qquad \text{(Lower-level)}$$

subject to

$$\begin{aligned} &\tilde{3}X_1 + \tilde{2}X_2 + X_3 + \tilde{3}X_4 \le \tilde{40} \\ &\tilde{2}X_1 + \tilde{4}X_2 + X_3 + \tilde{2}X_4 \le \tilde{35} \\ &X_1 + \tilde{2}X_2 + X_3 + \tilde{2}X_4 \le \tilde{30} \\ &X_1, X_2, X_3, X_4 \ge 0 \end{aligned} \qquad (77)$$

where all the fuzzy numbers are assumed as triangular fuzzy numbers and are given as follows:

$$\begin{aligned} &\tilde{2} = (0,2,3), \tilde{3} = (2,3,4), \tilde{4} = (3,4,5), \\ &\tilde{5} = (4,5,6), \tilde{6} = (5,6,7), \tilde{8} = (6, 8, 10), \\ &\tilde{9} = (8, 9, 10), 3\tilde{0} = (28,30,32), \\ &3\tilde{5} = (33,35,37), 4\tilde{0} = (35,40,45) \end{aligned}$$

By replacing the fuzzy coefficients by their α-cuts, problem (77) can be written as

$$\underset{X_1, X_2}{\text{Maximize}}\ {}^{\alpha}(Z_1)^U = (6-\alpha)X_1 + (7-\alpha)X_2 + (5-\alpha)X_3 + (3-\alpha)X_4$$

$$\underset{X_3, X_4}{\text{Maximize}}\ {}^{\alpha}(Z_2)^U = (10-2\alpha)X_1 + (10-\alpha)X_2 + (3-\alpha)X_3 + (5-\alpha)X_4$$

subject to

$(2+\alpha)$ X_1 $+2\alpha X_2$ $+X_3+$ $((2+\alpha)$ X_4 $\le 45\text{-}5\alpha$

2α $X_1+(3+\alpha)X_2$ $+X_3+2\alpha X_4$ $\le 37\text{-}2\alpha$

X_1 $+2\alpha X_2$ $+X_3+2\alpha X_4$ $\le 32\text{-}2\alpha$

X_1, X_2, X_3, X_4 ≥ 0

For, α = .5, let the decision makers provide the preference bounds to the decision variables

$0 \le X_1 \le 15$,

$0 \le X_2 \le 4$,

$0 \le X_3 \le 10$,

$0 \le X_2 \le 4$

Then FGP Model (1) offers the solution

λ^* =0.1220651, X_1^* =11.57139, X_2^* = 4, X_3^* = 9.571533, X_4^* = 0 with Z_1^* = 132.7145, Z_2^* =166.0713, μ_{Z_1} = 0.8779349, μ_{Z_2} = 0.8779349, L_1= 0.2441301, L_2 = 0.1726261, L_∞= 0.1220651

FGP Model (IIa) offers the solution

ζ^* = 0.1422829E-02, X_1^* =11.4, X_2^* = 4, X_3^* = 10, X_4^* = 0;

Z_1^* = 133.7, Z_2^* =165.6, μ_{Z_1} = 0.884454, μ_{Z_2} = 0.8754433, L_1= 0.2401026, L_2 = 0.1698977, L_∞=0.1245567.

FGP Model (IIb) offers the solution

ξ^* = 0.2401026, X_1^* =11.4, X_2^* = 4, X_3^* = 10, X_4^* = 0;

Z_1^* = 133.7, Z_2^* =165.6, μ_{Z_1} = 0.884454, μ_{Z_2} = 0.8754433, L_1= 0.2401026, L_2 = 0.1698977, L_∞=0.1245567.

Table1. Comparison of distances for the optimal solutions of example 1 based on FGP Models

Method	Z_1^*, Z_2^*	μ_{Z_1}, μ_{Z_2}	L_1	L_2	L_∞
FGP Model (I)	132.7145, 166.0713	.8779349, .8779349	.2441301	.1726261	.1220651
FGP Model (IIa)	133.7, 165.6	.884454, .8754433	.2401026	.1698977	.1245567
FGP Model (IIb)	133.7, 165.6	.884454, .8754433	.2401026	.1698977	.1245567

On comparing L_1, and L_2 we see that FGP model (IIa) and (IIb) offer better optimal solution. On comparing, L_∞ we see that FGP model (I) offers better optimal solution.

8. Conclusions

In this paper, FGP due to Pramanik and Roy [14, 16] is slightly modified and applied for solving BLPPs with fuzzy parameters. It is an alternative way to solve BLPPs with fuzzy parameters. Distance function is used to identify which FGP model offer better compromise optimal solution. The proposed approach can be extended to optimization problems in different areas, such as decentralized planning problems, agricultural planning problems and other real world multi-objective programming problems involving fuzzily described different parameters. The proposed approach can also be extended for multilevel multi-objective programming problem with fuzzy parameters.

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