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PRODUCTION PLANNING UNDER UNCERTAIN DEMANDS AND YIELDS

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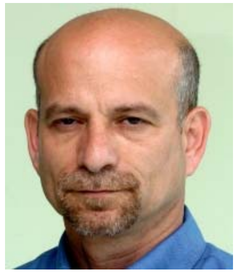

Abstract: *The periodic demands of a single product are forecasted and given by a distribution function for each period. The product can be manufactured in n plants with heterogeneous characters. Each plant has its specific stochastic production capability. The expected capability and the standard deviation of each plant can be increased by allocation of additional budgets. The problem is to determine the total budget needed and its distribution among the n plants in order to ensure a complete fulfillment of the demands according to the due dates and the pre-given confidence levels.*

Key words: *production planning; chance constrained; capability-cost trade-offs; random yield*

1. Introduction

The planning process of global production for a new product with numerous quantities addressed to anonymous customers (e.g., semiconductors, pharmaceutics, etc.) forces the corporation management to take the following principal decisions: 1) how much to produce, 2) where to produce, and 3) how to divide the production among a number of optional producers.

Mostly, actual demand fluctuates around the mean of demand distribution. Assuming that the mean of the underlying demand pattern is known, this fluctuation constitutes demand uncertainty. However, the expected demand can also vary through time, such as when seasonality is present. In such cases the true mean of the demand distribution is not stationary through time. Demand variability over time includes both demand uncertainty and variation due to the shifting mean of the demand distribution (Enns, 2002).

The presence of random yields can considerably complicate production planning and control. When the manufacturers control their inputs but the outputs exhibit random yields, coordination in such systems becomes quite complex. Two variants of demand have been addressed in the literature: 1) rigid demand - where an order must be satisfied in its entirety (possibly necessitating multiple manufacturing runs), and 2) non-rigid demand - where there is a penalty for a shortage (only one manufacturing run). The determination of monthly productions is particularly challenging when yields are random and demand needs to be satisfied in its entirety (i.e., rigid demand). The efficient planning of monthly productions often becomes a crucial economic factor. As a result the modeling of production with random yields has attracted the attention of many researchers (for a literature review see Yano and Lee 1995).

Random yield disables satisfaction of demand in its entirety, but determining strict chance constraint enables us to attain close claim to rigid demand. Laslo (2003) clarified that when additional budget is invested in order to obtain a rigid performance, we should refer to the impact of this act on the performance fractile and not on its impact on the expected performance. Such an approach puts the delivery objectives before the objective of reducing superfluous production.

Laslo and Gurevich (2007) have developed an iterative procedure for the minimization of budget that is required for executing the activities chain with chance constrained lead-time. The procedure assumes fixed coefficient variance while budget is added in order to increase the execution speed. This iterative procedure is applicable as well for the minimization of the total budget that should be allocated among heterogeneous plants (differing by initial investment, productivity and yield variance) which are supposed to supply together a rigid known demand under strict chance constraint. Laslo et al. (2009) have introduced another procedure that resolves problems where the optimization is carried out for several known rigid demands with a common due-date but under different chance constraints. They assumed for each producer a standard deviation of the yield that increases proportionally with the production and linearly with allocated budget.

This paper introduces a solution for a comprehensive problem of operating manufacturing with heterogeneous plants that differ by their investment-capacity tradeoff curves and their yield distributions. We consider: 1) monthly rigid demands (i.e., several orders with different due days given as a time series), 2) uncertain nonnegative demands with different expected amount and different variance of demand, and 3) random yield. The objective is to establish a global production plan that minimizes the total investment in the production plants, subject to monthly rigid deliveries and under pre-given chance constraints.

2. Notation

Let us introduce the following terms:

$\{j\}$ - an index for the months, $j = 1,2,...,k$;

O^j - the j's monthly demand, $j = 1,2,...,k$ (a random variable with known distribution);

$(1-\alpha_j)$	-	the lower bound probability (confidence level) for complete fulfillment of the monthly demand O_j, $0<\alpha_j<0.5$, $j=1,2,...,k$;
t_j	-	the time for supplying of the monthly demand O_j, $t_1 \le t_2 \le ... \le t_k$;
$\{i\}$	-	an index for the plants, $i=1,2,...,n$;
p_i	-	the normal production quantity of plant i up to the lead time t_k (a random variable with known expectation $E(p_i)$), given that the normal budget $c_{E(p_i)}$ was allocated for plant i;
$c_{E(p_i)}$	-	the known deterministic budget that enables a normal production quantity p_i, at plant i for the planning horizon $[t_1,t_k]$;
$\sigma(p_i)$	-	the known standard deviation of p_i;
P_i	-	the crash production quantity of plant i up to the lead time t_k (a random variable with known expectation $E(P_i)$), given that the crash budget $c_{E(P_i)}$ was allocated for plant i;
$c_{E(P_i)}$	-	the known deterministic budget that enables the crash production quantity P_i (capital P), at plant i for the planning horizon $[t_1,t_k]$;
q_i^k	-	the production quantity of plant i for the planning horizon $[t_1,t_k]$ (a random variable with expected value $E(q_i^k)$, $E(p_i) \le E(q_i^k) \le E(P_i)$, that is dependent on the deterministic budget c_i allocated to the plant i);
c_i	-	the budget (a decision variable) that enables q_i^k production quantity of plant i for the planning horizon $[t_1,t_k]$, $c_{E(p_i)} \le c_i \le c_{E(P_i)}$;
$\underline{c}=(c_1,...,$	-	a vector of the distributed budget among all plants.
C	-	the total budget allocated to all plant: $C=\sum_{i=1}^{n} c_i$;
Q^k	-	the total production for the planning horizon $[t_1,t_k]$ (a random variable), $Q^k=\sum_{i=1}^{n} q_i^k$;
Q_α^k	-	the α quintile of Q^k's distribution, $0<\alpha<0.5$;
Q^j	-	the total production at the horizon $[t_1,t_k]$; $Q^j=\sum_{i=1}^{n} q_i^j$, $j=1,2,...,k$.

3. Problem Definition

We consider n plants (production units) that can produce the same product. Each plant i, $i=1,2,...,n$ has a stochastic production capability q_i^k and needs a deterministic budget c_i, $c_{E(p_i)} \le c_i \le c_{E(P_i)}$, in order to activate the production capability q_i^k for the planning horizon $[t_1,t_k]$.

We assume that the relation for the expected production capability, given that c_i budget was allocated for plant i, $E(q_i^k|c_i)$, is given by a continuous linear increasing curve:

$$E(q_i^k|c_i) = \varphi_i c_i + \gamma_i \, , \tag{1}$$

where $\varphi_i = \dfrac{E(P_i) - E(p_i)}{c_{E(P_i)} - c_{E(p_i)}}$, $\gamma_i = \dfrac{c_{P_i} E(p_i) - c_{p_i} E(P_i)}{c_{E(P_i)} - c_{E(p_i)}}$.

We also assume that the randomness in the production capability of plant i is realized only once, immediately after the investment of c_i, $i = 1,2,...,n$. Therefore the production quantity of the plant i from the beginning of the production and until the time t_j, $j = 1,2,...,k$, $i = 1,2,...,n$ is defined according to the following equation (2).

$$q_i^j = \frac{t_j}{t_k} q_i^k . \tag{2}$$

Therefore, for all $j = 1,2,...,k$, $i = 1,2,...,n$ we have

$$E(q_i^j|c_i) = \frac{t_j}{t_k}(\varphi_i c_i + \gamma_i). \tag{3}$$

In addition, we assume a normal distribution of the total output Q^j, $j = 1,2,...,k$ of all n plants, statistical independence among the plants and nonnegativity, i.e. $E(p_i) - 4\sigma(p_i) > 0$, $E(q_i^j) - 4\sigma(q_i^j) > 0$ for $i = 1,2,...,n$ and $j = 1,2,...,k$. We emphasize that despite this assumption, it is not necessary to assume any specific distribution for the random variables P_i, p_i, q_i^j, $i = 1,2,...,n$, $j = 1,2,...,k$ since normality of the random variables Q^j can be justified by its definition together with the Central Limit Theorem.

Following Laslo (2003) we presume a fixed coefficient variance (FCV) model. This model assumes that the expected production quantity and the production quantity's standard deviation are both affected by additional budget, but the production coefficient variance is constant for any budget c_i and in any time:

$$K_i = \frac{E(q_i^k|c_i)}{\sigma(q_i^k|c_i)} = \frac{E(p_i)}{\sigma(p_i)}, \; i = 1,...,n, \tag{4}$$

in other words, wherever the average performance is increased, the standard deviation is also increased and at the same rate.

By assumptions (2) and (4) for all $j = 1,2,...,k$, $i = 1,2,...,n$ we have

$$K_i = \frac{E(q_i^j|c_i)}{\sigma(q_i^j|c_i)} = \frac{E(p_i)}{\sigma(p_i)}. \tag{5}$$

Finally we assume that for each point of time t_j, $j = 1,2,...,k$, a new delivery order O^j with random demand is set for the product. Hence there are $k \geq 1$ stochastic delivery orders for the product of the plants. An order O^j is a random variable with known distribution and must be supplied with probability of at least $(1-\alpha_j)$, $j=1,2,...,k$. The delivery order O^j does not depend on the production quantities of plants. We study here in details the case where for all $j=1,2,...,k$, the distribution of the O^j are normal distributed with known expected value and variance. For other situations (non-normal distribution for the delivery orders O^j) the analysis can be more complex but is based on similar considerations.

The main objective of the problem is to find the minimal budget and its distribution among all plants in order to ensure the fulfillment of all orders subject to the required probabilities.

4. The Solution

For all $j=1,2,...,k$ we need to fulfil the following inequalities

$$P\left(Q^j > O^1+...+O^j \middle| \underline{c}\right) \geq 1-\alpha_j \ , \ j=1,2,...,k,$$

or equivalently

$$P\left(Q^j - \sum_{m=1}^{j} O^m > 0 \middle| \underline{c}\right) \geq 1-\alpha_j \ , \ j=1,2,...,k \ . \qquad (6)$$

Since Q^j and $\sum_{m=1}^{j} O^m$ are independent normally distributed random variables, the random variable $Q^j - \sum_{m=1}^{j} O^m$ also has a normal distribution with the following expectation and variance:

$$E\left(Q^j - \sum_{m=1}^{j} O^m \middle| \underline{c}\right) = \sum_{i=1}^{n} E\left(q_i^j \middle| \underline{c}\right) - \sum_{m=1}^{j} E\left(O^m\right) = \frac{t_j}{t_k}\sum_{i=1}^{n}(\varphi_i c_i+\gamma_i) - \sum_{m=1}^{j} E\left(O^m\right),$$

$$V\left(Q_j - \sum_{m=1}^{j} O^m \middle| \underline{c}\right) = \sum_{i=1}^{n} V\left(q_i^j \middle| \underline{c}\right) + \sum_{m=1}^{j} V\left(O^m\right) = \left(\frac{t_j}{t_k}\right)^2 \sum_{i=1}^{n} \frac{(\varphi_i c_i+\gamma_i)^2}{K_i^2} + \sum_{m=1}^{j} V\left(O^m\right).$$

By (6) we get $\left(Q^j - \sum_{m=1}^{j} O^m\right)_{\alpha_j} \geq 0$, where $\left(Q^j - \sum_{m=1}^{j} O^m\right)_{\alpha_j}$ is the α_j quintile of the $Q^j - \sum_{m=1}^{j} O^m$ distribution. Straightforwardly we have:

$$\left(Q^j - \sum_{m=1}^{j} O^m\right)_{\alpha_j} = \frac{t_j}{t_k}\sum_{i=1}^{n}(\varphi_i c_i+\gamma_i) - \sum_{m=1}^{j} E\left(O^m\right) + Z_{\alpha_j}\sqrt{\left(\frac{t_j}{t_k}\right)^2 \sum_{i=1}^{n} \frac{(\varphi_i c_i+\gamma_i)^2}{K_i^2} + \sum_{m=1}^{j} V\left(O^m\right)}, \qquad (7)$$

where Z_{α_j} is the α_j quintile of the normal standard distribution.

Therefore, by (6), (7), the solution for the problem is equivalent finding the vector of optimal budgets $\underline{c}=(c_1,...,c_n)$ that minimizes the total budget $C=\sum_{i=1}^{n} c_i$, subject to:

$$\frac{t_j}{t_k}\sum_{i=1}^{n}(\varphi_i c_i+\gamma_i)+Z_{\alpha_j}\sqrt{\left(\frac{t_j}{t_k}\right)^2\sum_{i=1}^{n}\frac{(\varphi_i c_i+\gamma_i)^2}{K_i^{\,2}}+\sum_{m=1}^{j}V(O^m)}\geq\sum_{m=1}^{j}E(O^m),\quad j=1,2,\dots,k, \qquad (8)$$

and subject to the budget constraints:

$$c_{E(p_i)}\leq c_i\leq c_{E(P_i)} \quad , \quad i=1,\dots,n$$

The following inequality (9) guarantees that any additional budget in each plant i, $i=1,2,\dots,n$ increases the probability that the total production will fulfil the cumulative delivery order constraints until time t_j for all $j=1,2,\dots,k$.

Statement 1. If

$$\sqrt{\sum_{i=1}^{n}\left(\frac{\varphi_i c_i+\gamma_i}{K_i}\right)^2+\left(\frac{t_k}{t_j}\right)^2\sum_{m=1}^{j}V(O^m)}\geq -Z_{\alpha_j}\left(\frac{\varphi_i c_i+\gamma_i}{K_i^{\,2}}\right), \qquad (9)$$

for all $i=1,2,\dots,n$, $j=1,2,\dots,k$, then the quintile $\left(Q^j-\sum_{m=1}^{j}O^m\right)_{\alpha_j}$ is an increasing function of c_i for all $i=1,2,\dots,n$, $j=1,2,\dots,k$.

The following *Proposition 1* provides the necessary and sufficient conditions for the existence of a unique solution for the considered problem.

Proposition 1.

If for all $i=1,2,\dots,n$, $j=1,2,\dots,k$, equation (9) holds then the considered problem has a unique solution if and only if for all $j=1,2,\dots,k$

$$\frac{t_j}{t_k}\sum_{i=1}^{n}E(P_i)+Z_{\alpha_j}\sqrt{\left(\frac{t_j}{t_k}\right)^2\sum_{i=1}^{n}\left(\frac{E(P_i)}{K_i}\right)^2+\sum_{m=1}^{j}V(O^m)}\geq\sum_{m=1}^{j}E(O^m). \qquad (10)$$

Proposition 1, i.e., inequalities (9) and (10) guarantee the existence of a unique optimal solution for the considered problem.

In order to solve the considered chance-constrained programming problem, we can solve its certainty equivalent as a mathematical programming problem as defined by (8). But since the total budget is bounded: $\sum_{i=1}^{n}c_{E(p_i)}\leq C\leq\sum_{i=1}^{n}c_{E(P_i)}$ and since in any real life problem a budget is not a continuous entity, the budget can be considered as if it has a finite number of alternative values. Hence, after verifying by (9), (10) the existence and uniqueness of the optimal solution, one can attain it by examination of all the finite integer possibilities for budget allocation (dollars or cents), satisfying the constraints. Alternatively, the optimal solution can be obtained by optimization software package.

Remark 1.

A deterministic delivery order O^j can be considered as a "normal" random variable with expectation $E(O^j)=O^j$ and variance $V(O^j)=0$. Therefore the case where all the delivery orders are deterministic is only a special case of the considered problem and our analysis is also valid for this case.

5. Numerical Examples

We consider a situation with 3 plants and 2 monthly demands: $n = 3$, $k = 2$.

The upper bound probabilities α_j for full supplying of demands O^j, $j = 1,2$ are: $\alpha_1 = 0.001$, $\alpha_2 = 0.025$. That is, $Z_{0.001} = -3.09023$, $Z_{0.05} = -1.95996$.

Also given:

$E(p_1) = 25.00$, $E(P_1) = 220.00$, $c_{E(p_1)} = 75.00$, $c_{E(P_1)} = 250.00$, $\sigma(p_1) = 8.00$,

$E(p_2) = 50.00$, $E(P_2) = 250.00$, $c_{E(p_2)} = 100.00$, $c_{E(P_2)} = 350.00$, $\sigma(p_2) = 2.00$,

$E(p_3) = 50.00$, $E(P_3) = 200.00$, $c_{E(p_3)} = 25.00$, $c_{E(P_3)} = 450.00$, $\sigma(p_3) = 5.00$,

$t_1 = 50$, $t_2 = 100$.

Then we have:

$\varphi_1 = 1.11429$, $\gamma_1 = -58.57140$, $K_1 = \frac{25}{8} = 3.12500$,

$\varphi_2 = 0.80000$, $\gamma_2 = -30.00000$, $K_2 = 25.00000$,

$\varphi_3 = 0.35294$, $\gamma_3 = 41.17650$, $K_3 = 10.00000$.

First we consider a situation with deterministic monthly demands: $O^1 = 200.00$, $O^2 = 150.00$. That is, by Remark 1, $E(O^j) = O^j$, $V(O^j) = 0$, $j = 1,2$.

By a straightforward calculation we find that the equations (9), (10) are valid for this example. Therefore by Proposition 1 there is a unique optimal solution of the considered problem. By examination of all the finite integer possibilities for budget allocation, satisfying the constraints (8), we get this optimal vector of the budget allocation among all plants:

$$(c_1, c_2, c_3) = (114.70, 350.00, 373.37),$$

and the total optimal (minimal) budget allocated is $C = \sum_{i=1}^{3} c_i = 838.07$.

Secondly we consider the same situation as in the previous case, but with stochastic normal distributed demands such that: $O^1 \sim N(200, 20^2)$, $O^2 \sim N(150, 15^2)$.

By a straightforward calculation we find that the equations (9), (10) are valid for this example too. Therefore by Proposition 1 there is a unique optimal solution for the considered problem. By examination of all the finite integer possibilities for budget allocation, satisfying the constraints (8), we get this optimal vector of budget allocation among all factories:

$$(c_1, c_2, c_3) = (204.42, 350.00, 450.00),$$

the total optimal (minimal) budget allocated to all factories is $C = \sum_{i=1}^{3} c_i = 1,004.42$.

Finally, we consider the same situation as in the previous case, but with stochastic uniform distributed delivery orders such that: $O^1 \sim Uni(170,230)$, $O^2 \sim Uni(125,175)$.

Based on analysis which is similar to that presented for two previous examples, and by examination of all the finite integer possibilities for budget allocation we find that the optimal vector of the budget allocation among all plants:

$$(c_1, c_2, c_3) = (148.98,\ 350.00,\ 450.00),$$

the total optimal (minimal) budget allocated to all plants is $C = \sum_{i=1}^{3} c_i = 948.98$.

6. Summary and Conclusions

This paper gives a comprehensive analysis for the problem and a procedure that can help management to solve it, i.e., to determine how much budget is needed and how to distribute the budget among the plants in order to increase its capabilities and to guarantee the fulfillment of the orders under some chance constraints.

The first step is to verify that the problem has a feasible solution. This can be done by Proposition 1 that states roughly that the expected total capabilities of all plants must be sufficiently larger than the total cumulative orders at any time. We have proved that if the allocation of the crashed budgets $c_{E(P_i)}$ to each plant $i = 1,...,n$ is a feasible solution, then the problem has a unique optimal solution. The optimal solution can be obtained by a discrete search among the bounded budget or by optimization software package.

Although we assumed normal distributions for all the random variables, we demonstrated that even if the order quantities have non normal distributions, the considered problem can be solved in a similar way. Solutions for normal distributions and uniform distributions were presented through numerical examples.

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