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STRUCTURE DECISION MAKING BASED ON UNIVERSAL GENERATING FUNCTIONS FOR REFRIGERATION SYSTEM

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Abstract: *This paper presents a method for calculation of reliability measures for supermarket refrigeration system. The system and its components can have different performance levels ranging from perfect functioning to complete failure and, so it can be interpreted as a multi-state system. Calculated reliability measures are used for decision making of system structure. The suggested approach based on combined Universal Generating Functions and stochastic processes method for computation of availability, output performance and performance deficiency for multi-state system. Corresponding procedures are suggested. A numerical example is presented in order to illustrate the approach.*

***Key words**: reliability measures; multi-state system; Universal Generating Functions; availability; output performance; performance deficiency*

1. Introduction

Supermarkets suffer serious financial losses owing to problems with their refrigeration systems. A typical supermarket may contain more than one hundred individual refrigerated cabinets, cold store rooms and items of plant machinery which interact as part of a complex integrated refrigeration system within the store. Things very often go wrong with individual units (icing up of components, electrical or mechanical failure, and so forth…) or with components which serve a network of units (coolant tanks, pumps, compressors, and so on).

The most commonly used refrigeration system for supermarkets today is the multiplex direct expansion system (Baxter (2002), IEA Annex 26 (2003)). All display cases and cold store rooms use direct expansion air-refrigerant coils that are connected to the system compressors in a remote machine room located in the back or on the roof of the store. Heat rejection is usually done with air-cooled condensers with simultaneously working axial blowers mounted outside. Multiple compressors are mounted on a skid, or rack, and are piped with common suction and discharge refrigeration lines. Using multiple compressors in parallel provides a means of capacity control, since the compressors can be selected and cycled as needed to meet the refrigeration load.

Due to the system's highly integrated nature, a fault in a single unit or item of machinery can't have detrimental effects on the entire store, only decrease of system cool capacity. Failure of compressor or axial condenser blower leads to partial system failure (degradation of output cooling capacity) as well as to complete failures of the system. We treat refrigeration system as multi-state system (MSS), where components and systems have an arbitrary finite number of states. According to the generic MSS model (Lisnianski and Levitin 2003), the system can have different states corresponding to the system's performance rates. The performance rate of the system at any instant is a discrete-state continuous-time stochastic process .

In this paper, a generalized approach (Lisnianski, 2004), (Lisnianski, 2007) was extended and applied for decision making for multi-state supermarket refrigeration system structure. The approach is based on the combined Universal Generating Functions UGF) and stochastic processes method for computation of availability, output performance and performance deficiency for multi-state system.

2. The Method Description

2.1. Performance Stochastic Process for Multi-state Element

In general case any element j in MSS can have k_j different states corresponding to different performance, represented by the set $\mathbf{g}_j = \left\{ g_{j1}, ..., g_{jk_j} \right\}$, where g_{ji} is the performance rate of element j in the state i, $i \in \left\{ 1, 2, ..., k_j \right\}$.

At first stage in according to the suggested method a model of stochastic process should be built for each multi-state element in MSS. Based on this model state probabilities

$$p_{ji}(t) = \Pr\{G_j(t) = g_{ji}\},\ i \in \{1, ..., k_j\},$$

for every MSS's element $j \in \left\{ 1, ..., n \right\}$ can be obtained. These probabilities define output stochastic process $G_j(t)$ for each element j in the MSS.

At the next stage the output performance distribution for the entire MSS at each time instant t should be defined based on previously determined states probabilities for all elements and system structure function. At this stage UGF technique provides simple procedure that is based only on algebraic operation.

Without loss of generality here we consider a multi-state element with minor failures and repairs.

2.2. Markov Model for Multi-state Element

If all times to failures and repair times are exponentially distributed the performance stochastic process will have Markov property and can be represented by Markov model. Here for the simplicity we omit index *j* and assume that element has *k* different states as presented in the Fig. 1. For Markov process each transition from the state *s* to any state *m*, (*s, m*=1,…,*k*) has its own associated transition intensity that will be designated as a_{sm}. In our case any transition is caused by element's failure or repair. If *m*<*s*, then $a_{sm} = \lambda_{sm}$, where λ_{sm} is a failure rate for the failures that cause element transition from state *s* to state *m*. If *m*>*s*, then $a_{sm} = \mu_{sm}$, where μ_{sm} is a corresponding repair rate. With each state *s* the corresponding performance g_s is associated.

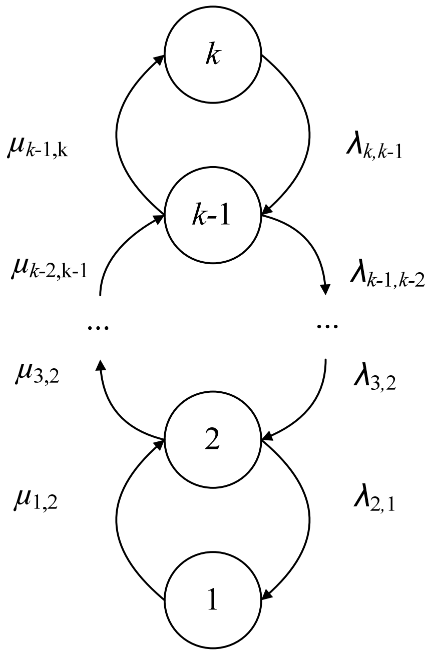

Figure 1. State-space diagram for Markov model of repairable Multi-state element

Let $p_s(t), s=1,\ldots,k$ be the state probabilities of element's performance process $G(t)$ at time *t*: $p_s(t) = \Pr\{G(t) = g_s\},\ s=1,\ldots,k;\ t \ge 0.$

The following system of differential equations for finding the state probabilities $p_s(t), s=1,\ldots,k$ for the homogeneous Markov process can be written

$$\frac{dp_s(t)}{dt} = \left[\sum_{\substack{i=1\\i\neq s}}^{k} p_i(t)a_{is}\right] - p_s(t)\sum_{\substack{i=1\\i\neq s}}^{k} a_{si}\ . \qquad (1)$$

In our case all transitions are caused by element's failures and repairs. So, corresponding transition intensities a_{is} are expressed by the element's failure and repair rates. Therefore, the corresponding system of differential equations may be written

$$\begin{aligned}
&\frac{dp_1(t)}{dt} = -\mu_{12}p_1(t) + \lambda_{21}p_2(t) \\
&\frac{dp_2(t)}{dt} = \mu_{12}p_1(t) - (\lambda_{21} + \mu_{23})p_2(t) + \lambda_{32}p_3(t) \\
&\ldots \\
&\frac{dp_k(t)}{dt} = \mu_{k-1,k}p_{k-1}(t) - \lambda_{k,k-1}p_k(t)
\end{aligned} \quad (2)$$

We assume that initial state will be the state *k* with best performance. Therefore, by solving the system (2) of differential equations under initial conditions $p_k(0)=1,\ p_{k-1}(0)=\ldots=p_2(0)=p_1(0)=0$, the states probabilities $p_s(t), s=1,\ldots,k$ can be obtained.

2.3. UGF for Multi-state System Reliability Evaluation

The generic MSS model consists of the performance stochastic processes $G_j(t)\in \mathbf{g}_j,\ j=1,\ldots,n$ for each system element *j*, and the system structure function that produces the stochastic process corresponding to the output performance of the entire MSS: $G(t)=f(G_1(t),...,G_n(t))$. At the previous stage all stochastic processes $G_i(t)$, $j=1,2,\ldots,n$ were completely defined by output performance distribution at any instant *t* for each system element.

In a traditional binary-state reliability interpretation (Modarres et al 1999) a reliability block diagram shows the interdependencies among all elements. The purpose is to show, by concise visual shorthand, the various block combinations (paths) that result in system success. Each block of the reliability block diagram represents one element of function contained in the system. All blocks are configured in series, parallel, standby, or combinations thereof as appropriate. The blocks in the diagram follow a logical order which relates the sequence of events during the prescribed operation of the system. The reliability model consists of a reliability block diagram and an associated mathematical or simulation model.

In a multi-state interpretation each block of the reliability block diagram represents one multi-state element of the system. A logical order of the blocks in the diagram is defined by the system structure function $f(G_1(t),...,G_n(t))$ as well as each block's *j* behavior is defined by the corresponding performance stochastic process $G_j(t)$.

At this stage based on previously determined output stochastic processes $G_j(t)$ for all elements $j=1,2,\ldots, n$, and on the given system structure function $f(G_1(t),...,G_n(t))$, an output performance stochastic process *G*(*t*) for the entire MSS should be defined $G(t)=f(G_1(t),...,G_n(t))$. It may be done by using UGF method.

At first, individual universal generating function (UGF) for each element should be written. For each element *j* it will be UGF $u_j(z,t)$ associated with corresponding stochastic processes $G_j(t)$. Then by using composition operators over UGF of individual elements and their combinations in the entire MSS structure, one can obtain the resulting UGF *U*(z,*t*) associated with output performance stochastic process *G*(*t*) of the entire MSS by using simple algebraic operations. This UGF *U*(z,*t*) defines the output performance distribution for the entire MSS at each time instant *t*. MSS reliability measures can be easily derived from this output performance distribution.

The following steps should be executed:

1. Having performances g_{ji} and corresponding probabilities $p_{ji}(t)$ for each element *j*, $j=1,...n,\ i=1,...k_j$, one can define UGF $u_j(z,t)$ associated with output performance stochastic process for this element in the following form:

$$u_j(z,t) = p_{j1}(t) z^{g_{j1}} + p_{j2}(t) z^{g_{j2}} + \ldots + p_{jk_j}(t) z^{g_{jk_j}} \qquad (3)$$

2. The composition operators Ω_{fser} (for elements connected in series), Ω_{fpar} (for elements connected in parallel) and $\Omega_{fbridge}$ (for elements connected in bridge structure) should be applied over the UGF of individual elements and their combinations. These operators one can find in (Lisnianski and Levitin, 2003), (Levitin, 2005), where corresponding recursive procedures for their computation were introduced for different types of systems. Based on these procedures the resulting UGF for the entire MSS can be obtained:

$$U(z,t) = \sum_{i=1}^{K} p_i(t) z^{g_i} \qquad (4)$$

where *K* is the number of entire system states and g_i is the entire system performance in the corresponding state *i*, *i*=1, ...,*K*.

3. Applying the operators $\delta_A, \delta_E, \delta_D$ over the resulting UGF of the entire MSS one can obtain the following MSS reliability indices:

– MSS availability *A*(*t*, *w*) at instant *t*>0 for arbitrary constant demand *w*

$$A(t,w) = \delta_A(U(z,t),w) = \delta_A(\sum_{i=1}^{K} p_i(t) z^{g_i}, w) = \sum_{i=1}^{K} p_i(t) 1(g_i - w \geq 0). \qquad (5)$$

– MSS expected output performance at instant t>0

$$E(t) = \delta_E(U(z,t)) = \delta_E(\sum_{i=1}^{K} p_i(t) z^{g_i}) = \sum_{i=1}^{K} p_i(t) g_i. \qquad (6)$$

– MSS expected performance deficiency at t>0 for arbitrary constant demand w

$$D(t,w) = \delta_D(U(z),w) = \delta_D(\sum_{i=1}^{K} p_i(t) z^{g_i}, w) = \sum_{i=1}^{K} p_i(t) \cdot \max(w - g_i, 0). \qquad (7)$$

3. Numerical Example

Consider the refrigeration system used in one of the Israel supermarkets (Frenkel et al. 2010). The system consists of 2 elements: 4 compressors, situated in the machine room and 2 main axial condenser blowers. Structure scheme of the system is presented in Fig.2. It is possible to add one additional blower. In this case the structure scheme of the system is presented in Fig.4.

3.1. System with 2 Condenser Blowers

Series-parallel refrigerating multi-state system with two blowers is presented in Figure 2. State-space diagram of the elements of this system is presented in Figure 3.

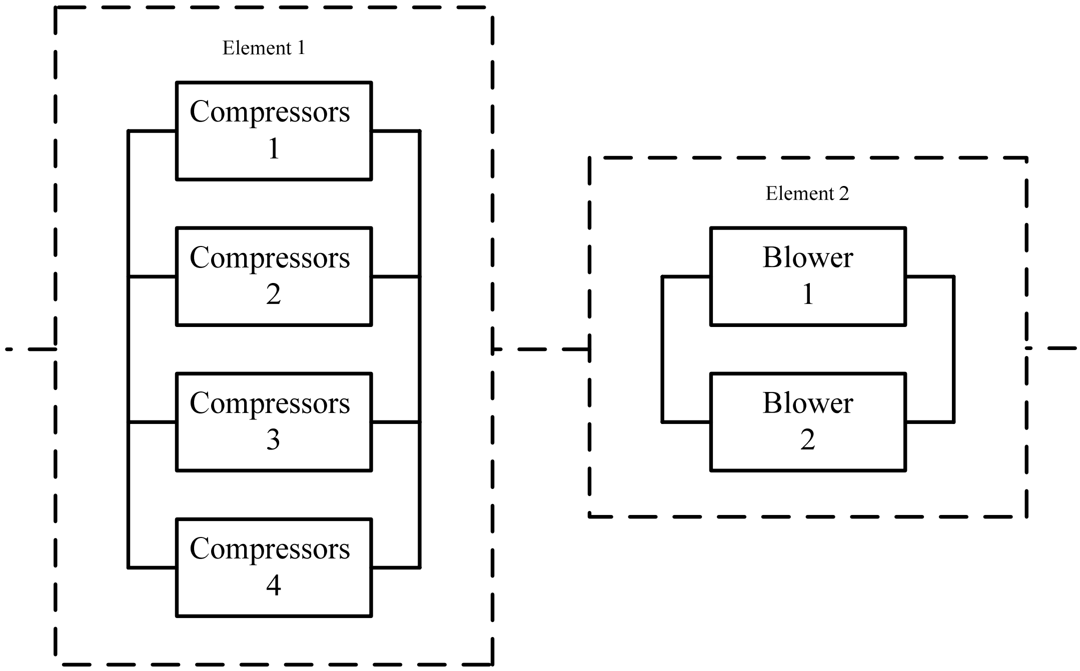

Figure 2. Series-parallel refrigerating multi-state system with two blowers

The performance of the elements is measured by their produce cold capacity (BTU per year). Times to failures and times to repairs are distributed exponentially for all elements. Elements are repairable. It is possible only minimal repair. Both elements are multi-state elements with minor failures and minor repairs. The first element can be in one of five states: a state of total failure corresponding to a capacity of 0, states of partial failures corresponding to capacities of $2.6 \cdot 10^9$, $5.2 \cdot 10^9$, $7.9 \cdot 10^9$ BTU per year and a fully operational state with a capacity of $10.5 \cdot 10^9$ BTU per year. For simplification we will present system capacity in 10^9BTU per year units. Therefore,

$$G_1(t) \in \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\} = \{0, 2.6, 5.2, 7.9, 10.5\}. \tag{8}$$

The failure rates and repair rates corresponding to the first element are $\lambda^C = 1 year^{-1}$, $\mu^C = 12 year^{-1}$.

The second element can be in one of three states: a state of total failure corresponding to a capacity of 0, state of partial failure corresponding to capacity of $5.2 \cdot 10^9$ BTU per year and a fully operational state with a capacity of $10.5 \cdot 10^9$ BTU per year. Therefore,

$$G_2(t) \in \{g_{21}, g_{22}, g_{23}\} = \{0, 5.2, 10.5\}. \tag{9}$$

The failure rate and repair rate corresponding to the second element are $\lambda^B = 10 year^{-1}$, $\mu^B = 365 year^{-1}$.

The MSS structure function is:

$$G_s(t)=f\left(G_1(t),G_2(t)\right)=\min\left\{G_1(t),G_2(t)\right\}. \tag{10}$$

The demand is constant: $w=5.0\cdot 10^9$ BTU per year.

Using combined UGF and stochastic process method we will find MSS availability $A(t,w)$, expected output performance $E(t)$ and expected performance deficiency *D*(*t*,*w*).

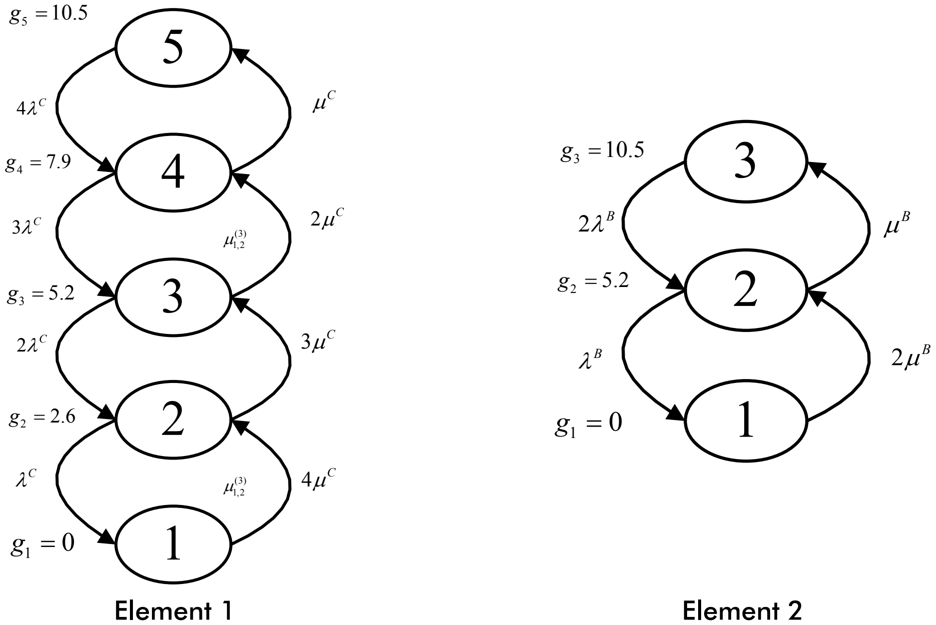

Figure 3. State-space diagram of the multi-state system with two blowers

Applying the described above two-stage procedure, we proceed as follows.

According to the Markov method we build the following systems of differential equations for each element separately (using the state-space diagrams presented in Figure 3).

- For element 1:

$$\begin{cases} \dfrac{dp_{11}(t)}{dt}=-4\mu^C p_{11}(t)+\lambda^C p_{12}(t) \\ \dfrac{dp_{12}(t)}{dt}=4\mu^C p_{11}(t)-(\lambda^C+3\mu^C)p_{12}(t)+2\lambda^C p_{13}(t) \\ \dfrac{dp_{13}(t)}{dt}=3\mu^C p_{12}(t)-(2\lambda^C+2\mu^C)p_{13}(t)+3\lambda^C p_{14}(t) \\ \dfrac{dp_{14}(t)}{dt}=2\mu^C p_{13}(t)-(3\lambda^C+\mu^C)p_{14}(t)+4\lambda^C p_{15}(t) \\ \dfrac{dp_{15}(t)}{dt}=\mu^C p_{14}(t)-4\lambda^C p_{15}(t). \end{cases} \tag{11}$$

Initial conditions are: $p_{11}(0)=p_{12}(0)=p_{13}(0)=p_{14}(0)=0;\ p_{15}(0)=1..$

- For element 2:

$$\begin{cases} \frac{dp_{21}(t)}{dt} = -2\mu^B p_{21}(t) + \lambda^B p_{22}(t) \\ \frac{dp_{22}(t)}{dt} = 2\mu^B p_{21}(t) - (\lambda^B + \mu^B) p_{22}(t) + 2\lambda^B p_{23}(t) \\ \frac{dp_{23}(t)}{dt} = \mu^B p_{22}(t) - 2\lambda^B p_{23}(t). \end{cases} \quad (12)$$

Initial conditions are: $p_{21}(0) = p_{22}(0) = 0;\ p_{23}(0) = 1.$

A closed form solution can be obtained for each of these 2 systems of differential equations. All calculations were made using MATLAB®. Corresponding expressions for states probabilities are the following.

- For element 1:

$$\begin{aligned} p_{11}(t) &= 1/28561 - 4/28561\, e^{-13t} + 6/28561\, e^{-26t} - 4/28561\, e^{-39t} + 1/28561\, e^{-52t}, \\ p_{12}(t) &= 48/28561 - 140/28561\, e^{-13t} + 132/28561\, e^{-26t} - 36/28561\, e^{-39t} + 4/28561\, e^{-52t}, \\ p_{13}(t) &= 864/28561 - 1584/28561\, e^{-13t} + 582/28561\, e^{-26t} - 132/28561\, e^{-39t} + 6/28561\, e^{-52t}, \\ p_{14}(t) &= 6912/28561 - 5184/28561\, e^{-13t} + 1584/28561\, e^{-26t} - 140/28561\, e^{-39t} + 4/28561\, e^{-52t}, \\ p_{15}(t) &= 20736/28561 - 6912/28561\, e^{-13t} + 864/28561\, e^{-26t} - 48/28561\, e^{-39t} + 1/28561\, e^{-52t}. \end{aligned} \quad (13)$$

- For element 2:

$$\begin{aligned} p_{21}(t) &= 4/5625 + 4/5625\, e^{-750t} - 8/5625\, e^{-375t}, \\ p_{22}(t) &= 292/5625 - 8/5625\, e^{-750t} - 284/5625\, e^{-375t}, \\ p_{23}(t) &= 5329/5625 + 4/5625\, e^{-750t} + 292/5625\, e^{-375t}. \end{aligned} \quad (14)$$

Therefore, one obtains the following output performance stochastic processes:

– element 1: $\begin{cases} \mathbf{g}_1 = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\} = \{0, 2.6, 5.2, 7.9, 10.5\}, \\ \mathbf{p}_1(t) = \{p_{11}(t), p_{12}(t), p_{13}(t), p_{14}(t), p_{15}(t)\}; \end{cases}$

– element 2: $\begin{cases} \mathbf{g}_2 = \{g_{21}, g_{22}, g_{23}\} = \{0, 5.2, 10.5\}, \\ \mathbf{p}_2(t) = \{p_{21}(t), p_{22}(t), p_{23}(t)\}. \end{cases}$

Having the sets $\mathbf{g}_j$, $\mathbf{p}_j(t)$ for $j=1,2$ one can define for each individual element j the *u*-function associated with the element's output performance stochastic process:

$$\begin{aligned} u_1(z,t) &= p_{11}(t) z^{g_{11}} + p_{12}(t) z^{g_{12}} + p_{13}(t) z^{g_{13}} + p_{14}(t) z^{g_{14}} + p_{15}(t) z^{g_{15}} = \\ &\quad p_{11}(t) z^{0} + p_{12}(t) z^{2.6} + p_{13}(t) z^{5.2} + p_{14}(t) z^{7.9} + p_{15}(t) z^{10.5}, \\ u_2(z,t) &= p_{21}(t) z^{g_{21}} + p_{22}(t) z^{g_{22}} + p_{23}(t) z^{g_{23}} = \\ &\quad p_{21}(t) z^{0} + p_{22}(t) z^{5.2} + p_{23}(t) z^{10.5}. \end{aligned} \quad (15)$$

Using the composition operator $\Omega_{f_{ser}}$ for refrigerating MSS one obtains the resulting UGF for the entire series MSS

$$U(z,t)=\Omega_{f_{ser}}(u_1(z,t),u_2(z,t)). \tag{16}$$

In order to find the resulting UGF U (z,t) for elements 1 and 2 connected in series the operator $\Omega_{f_{ser}}$ applied to individual UGF $u_1(z,t)$ and $u_2(z,t)$.

$$\begin{aligned}
U(z,t)&=\Omega_{f_{ser}}\left(u_1(z,t),u_2(z,t)\right)=\\
&=\Omega_{f_{ser}}\left(p_{11}(t)z^0+p_{12}(t)z^{2.6}+p_{13}(t)z^{5.2}+p_{14}(t)z^{7.9}+p_{15}(t)z^{10.5},\right.\\
&\quad\left.p_{21}(t)z^0+p_{22}(t)z^{5.2}+p_{23}(t)z^{10.5}\right)=\\
&=p_{11}(t)p_{21}(t)z^0+p_{11}(t)p_{22}(t)z^0+p_{11}(t)p_{23}(t)z^0+\\
&+p_{12}(t)p_{21}(t)z^0+p_{12}(t)p_{22}(t)z^{2.6}+p_{12}(t)p_{23}(t)z^{2.6}+\\
&+p_{13}(t)p_{21}(t)z^0+p_{13}(t)p_{22}(t)z^{5.2}+p_{13}(t)p_{23}(t)z^{5.2}+\\
&+p_{14}(t)p_{21}(t)z^0+p_{14}(t)p_{22}(t)z^{5.2}+p_{14}(t)p_{23}(t)z^{7.9}.\\
&+p_{15}(t)p_{21}(t)z^0+p_{15}(t)p_{22}(t)z^{5.2}+p_{15}(t)p_{23}(t)z^{10.5}.
\end{aligned} \tag{17}$$

In the resulting UGF $U(z,t)$ the powers of z are found as minimum of powers of corresponding terms.

Taking into account that $p_{11}(t)+p_{12}(t)+p_{13}(t)+p_{14}(t)+p_{15}(t)=1$ and $p_{21}(t)+p_{22}(t)+p_{23}(t)=1$, one can simplify the last expression for $U(z,t)$ and obtain the resulting UGF associated with the output performance stochastic process **g**, **p**(t) of the entire MSS in the following form

$$U(z,t)=\sum_{i=1}^{5}p_i(t)z^{g_i} \tag{18}$$

where

$$\begin{aligned}
&g_1=0, && p_1(t)=p_{11}(t)+\left(1-p_{11}(t)\right)p_{21}(t),\\
&g_2=2.6\cdot10^9\ \text{BTU/year}, && p_2(t)=p_{12}(t)\left[p_{22}(t)+p_{23}(t)\right],\\
&g_3=5.2\cdot10^9\ \text{BTU/year}, && p_3(t)=\left[p_{13}(t)+p_{14}(t)+p_{15}(t)\right]p_{22}(t)+p_{13}(t)p_{23}(t),\\
&g_4=7.9\cdot10^9\ \text{BTU/year}, && p_4(t)=p_{14}(t)p_{23}(t),\\
&g_5=10.5\cdot10^9\ \text{BTU/year}, && p_5(t)=p_{15}(t)p_{23}(t).
\end{aligned}$$

These two sets

$$\mathbf{g}=\{g_1,g_2,g_3,g_4,g_5\}\ \text{and}\ \mathbf{p}(t)=\{p_1(t),p_2(t),p_3(t),p_4(t),p_5(t)\}$$

completely define output performance stochastic process for the entire MSS.

Based on resulting UGF $U(z,t)$ of the entire MSS, one can obtain the MSS reliability indices. The instantaneous MSS availability for the constant demand level $w=5.0\cdot10^9$ BTU per year

$$A(t)=\delta_A(U(z,t),w)=\delta_A(\sum_{i=1}^{5}p_i(t)z^{g_i},5)=$$
$$\sum_{i=1}^{5}p_i(t)1(F(g_i,5)\geq 0)=p_3(t)+p_4(t)+p_5(t). \tag{19}$$

The instantaneous mean output performance at any instant t>0

$$E(t)=\delta_E(U(z,t))=\sum_{i=1}^{5}p_i(t)g_i=2.6p_2(t)+5.2p_3(t)+7.9p_4(t)+10.5p_5(t). \tag{20}$$

The instantaneous performance deficiency $D(t)$ at any time t for the constant demand w=5.0·10^9 BTU per year:

$$D(t)=\delta_D(U(z),w)=\sum_{i=1}^{5}p_i(t)\cdot\max(5-g_i,0)=$$
$$=p_1(t)(5-0)+p_2(t)(5-2.6)=5p_1(t)+2.4p_2(t). \tag{21}$$

Calculated reliability indices $A(t)$, $E(t)$ and $D(t)$ are presented on the Figures 6-8.

Note that instead of solving the system of K=5*3=15 differential equations (as it should be done in the straightforward Markov method) here we solve just two systems. The further derivation of the entire system states probabilities and reliability indices is based on using simple algebraic equations.

3.2. System with 3 Condenser Blowers

To increase reliability level of the system Supermarket decided to add additional blower and our goal is to compare reliability indices in new structure. The new refrigerating system structure is presented in Figure 4. State-space diagram of the elements of this system is presented in Figure 5.

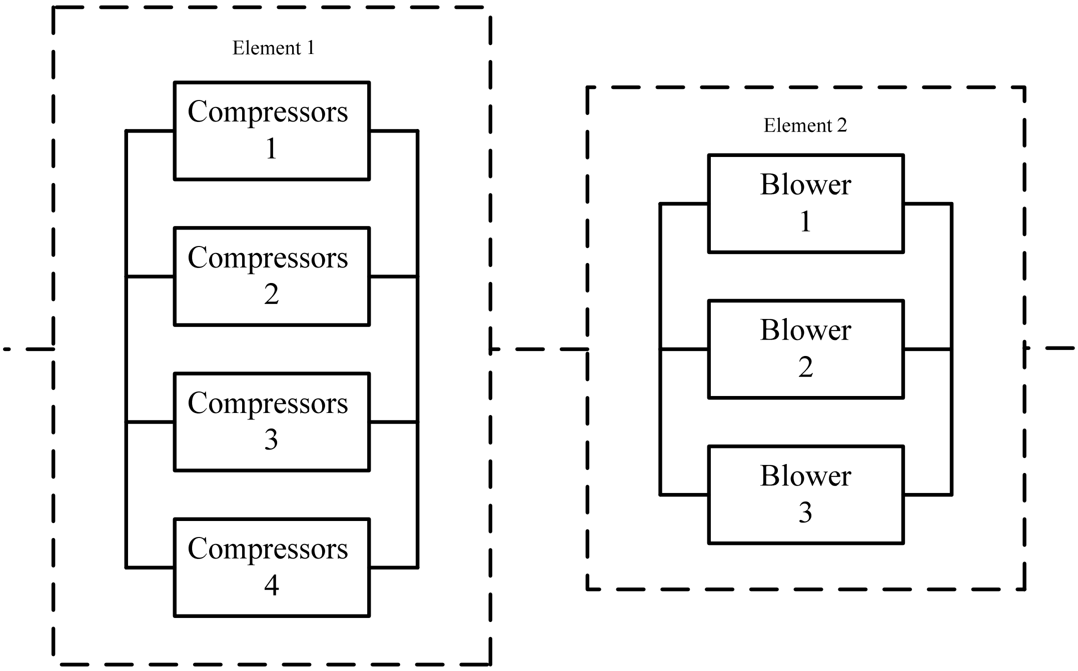

Figure 4. Series-parallel refrigerating multi-state system with 3 blowers

Like in previous case the system consists of two elements: block of 4 compressors and block of 3 blower. The performance of the elements is measured by their produce cold capacity (BTU per year). Times to failures and times to repairs are distributed exponentially for all elements. Elements are repairable. Both elements are multi-state elements with minor failures and minor repairs. The first element can be in one of five states: a state of total failure corresponding to a capacity of 0, states of partial failures corresponding to capacities of $2.6\cdot10^9$, $5.2\cdot10^9$, $7.9\cdot10^9$ BTU per year and a fully operational state with a capacity of $10.5\cdot10^9$ BTU per year. For simplification we will present system capacity in 10^9BTU per year units. Therefore,

$$G_1(t)\in\{g_{11},g_{12},g_{13},g_{14},g_{15}\}=\{0,2.6,5.2,7.9,10.5\}. \quad (22)$$

The failure rates and repair rates corresponding to the first element are $\lambda^C=1year^{-1}$, $\mu^C=12year^{-1}$.

The second element can be in one of 4 states: a state of total failure corresponding to a capacity of 0, state of partial failure corresponding to capacity of $5.2\cdot10^9$ BTU per year and two fully operational states with a capacity of $10.5\cdot10^9$ BTU per year. Therefore,

$$G_2^*(t)\in\{g_{21}^*,g_{22}^*,g_{23}^*,g_{24}^*\}=\{0,5.2,10.5,10.5\}. \quad (23)$$

The failure rate and repair rate corresponding to the second element are $\lambda^B=10year^{-1}$, $\mu^B=365year^{-1}$.

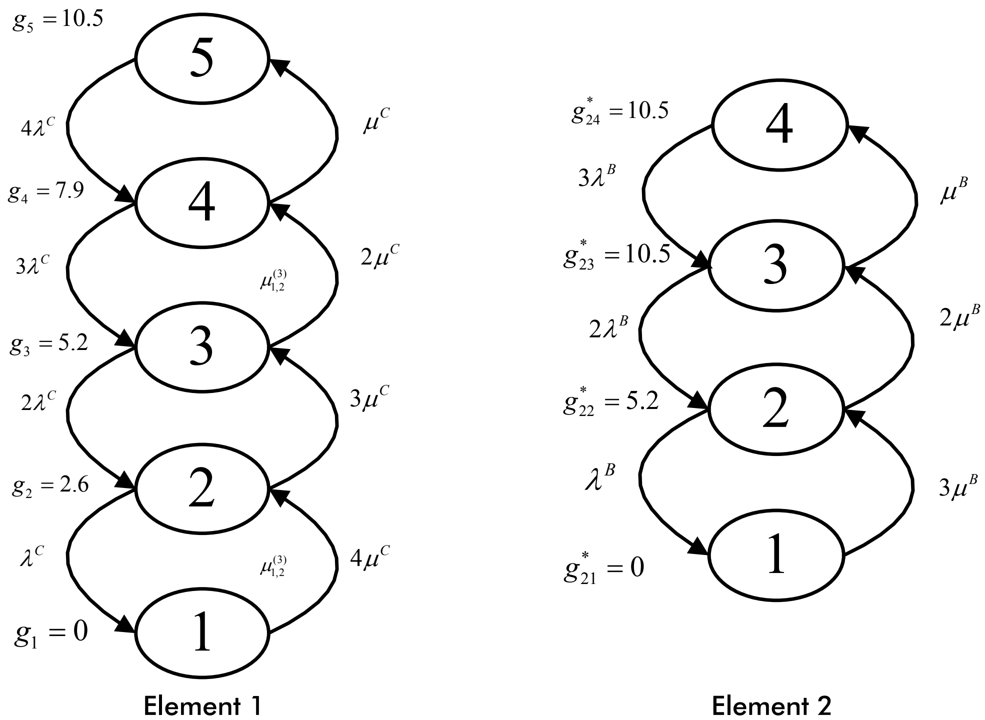

Figure 5. State-space diagram of the multi-state system with 3 blowers

The MSS structure function is:

$$G_s(t)=f\left(G_1(t),G_2^*(t)\right)=\min\{G_1(t),G_2^*(t)\}. \quad (24)$$

The demand is constant: w=$5.0\cdot10^9$ BTU per year.

Using combined UGF and stochastic process method we will find MSS availability $A(t,w)$, expected output performance $E(t)$ and expected performance deficiency *D*(*t*,*w*) for the system with additional blower.

Applying the described above two-stage procedure, we proceed as follows.

1. According to the Markov method we build the following systems of differential equations for each element separately (using the state-space diagrams presented in Figure 5).

For element 1 all calculations were proceeded earlier (13).

For element 2:

$$\begin{cases} \dfrac{dp_{21}^*(t)}{dt} = -3\mu^B p_{21}^*(t) + \lambda^B p_{22}^*(t) \\ \dfrac{dp_{22}^*(t)}{dt} = 3\mu^B p_{21}^*(t) - (\lambda^B + 2\mu^B) p_{22}^*(t) + 2\lambda^B p_{23}^*(t) \\ \dfrac{dp_{23}^*(t)}{dt} = 2\mu^B p_{22}^*(t) - (2\lambda^B + \mu^B) p_{23}^*(t) + 3\lambda^B p_{24}^*(t) \\ \dfrac{dp_{24}^*(t)}{dt} = \mu^B p_{23}^*(t) - 3\lambda^B p_{24}^*(t). \end{cases} \tag{25}$$

Initial conditions are: $p_{21}^*(0) = p_{22}^*(0) = p_{23}^*(0) = 0;\ p_{24}^*(0) = 1.$

A closed form solution can be obtained for the system of differential equations. Corresponding expressions for states probabilities are the following.

For element 2:

$$\begin{aligned} p_{21}^*(t) &= 8/421875 + 8/140625\, e^{-750t} - 8/140625\, e^{-375t} - 8/421875\, e^{-1125t}, \\ p_{22}^*(t) &= 292/140625 + 92/46875\, e^{-750t} - 64/15625\, e^{-375t} - 8/140625\, e^{-1125t}, \\ p_{23}^*(t) &= 10658/140625 - 64/15625\, e^{-750t} - 3358/46875\, e^{-375t} - 8/140625\, e^{-1125t}, \\ p_{24}^*(t) &= 389017/421875 + 292/140625\, e^{-750t} + 10658/140625\, e^{-375t} + 8/421875\, e^{-1125t}. \end{aligned} \tag{26}$$

Therefore, one obtains the following output performance stochastic processes:

– element 1: $\begin{cases} \mathbf{g}_1 = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\} = \{0, 2.6, 5.2, 7.9, 10.5\}, \\ \mathbf{p}_1(t) = \{p_{11}(t), p_{12}(t), p_{13}(t), p_{14}(t), p_{15}(t)\}; \end{cases}$

– element 2: $\begin{cases} \mathbf{g}_2^* = \{g_{21}^*, g_{22}^*, g_{23}^*, g_{24}^*\} = \{0, 5.2, 10.5, 10.5\}, \\ \mathbf{p}_2^*(t) = \{p_{21}^*(t), p_{22}^*(t), p_{23}^*(t), p_{24}^*(t)\}. \end{cases}$

1. Having the sets $\mathbf{g}_j$, $\mathbf{p}_j(t)$ for j=1,2 one can define for each individual element j the *u*-function associated with the element's output performance stochastic process:

$$\begin{aligned} u_1(z,t) &= p_{11}(t)z^{g_{11}} + p_{12}(t)z^{g_{12}} + p_{13}(t)z^{g_{13}} + p_{14}(t)z^{g_{14}} + p_{15}(t)z^{g_{15}} = \\ &\quad p_{11}(t)z^{0} + p_{12}(t)z^{2.6} + p_{13}(t)z^{5.2} + p_{14}(t)z^{7.9} + p_{15}(t)z^{10.5}, \\ u_2^*(z,t) &= p_{21}^*(t)z^{g_{21}^*} + p_{22}^*(t)z^{g_{22}^*} + p_{23}^*(t)z^{g_{23}^*} + p_{24}^*(t)z^{g_{24}^*} = \\ &\quad p_{21}^*(t)z^{0} + p_{22}^*(t)z^{5.2} + p_{23}^*(t)z^{10.5} + p_{24}^*(t)z^{10.5}. \end{aligned} \quad (27)$$

2. Using the composition operator $\Omega_{f_{ser}}$ for refrigerating MSS one obtains the resulting UGF for the entire series MSS

$$U(z,t) = \Omega_{f_{ser}}(u_1(z,t), u_2^*(z,t)). \quad (28)$$

In order to find the resulting UGF *U* (*z*,*t*) for elements 1 and 2 connected in series the operator $\Omega_{f_{ser}}$ applied to individual UGF $u_1(z,t)$ and $u_2(z,t)$.

$$\begin{aligned} U(z,t) &= \Omega_{f_{ser}}\left(u_1(z,t), u_2(z,t)\right) = \\ &= \Omega_{f_{ser}}\Big(p_{11}(t)z^{0} + p_{12}(t)z^{2.6} + p_{13}(t)z^{5.2} + p_{14}(t)z^{7.9} + p_{15}(t)z^{10.5}, \\ &\quad p_{21}^*(t)z^{0} + p_{22}^*(t)z^{5.2} + \left(p_{23}^*(t) + p_{24}^*(t)\right)z^{10.5}\Big) = \\ &= p_{11}(t)p_{21}^*(t)z^{0} + p_{11}(t)p_{22}^*(t)z^{0} + p_{11}(t)\left(p_{23}^*(t) + p_{24}^*(t)\right)z^{0} + \\ &+ p_{12}(t)p_{21}^*(t)z^{0} + p_{12}(t)p_{22}^*(t)z^{2.6} + p_{12}(t)\left(p_{23}^*(t) + p_{24}^*(t)\right)z^{2.6} + \\ &+ p_{13}(t)p_{21}^*(t)z^{0} + p_{13}(t)p_{22}^*(t)z^{5.2} + p_{13}(t)\left(p_{23}^*(t) + p_{24}^*(t)\right)z^{5.2} + \\ &+ p_{14}(t)p_{21}^*(t)z^{0} + p_{14}(t)p_{22}^*(t)z^{5.2} + p_{14}(t)\left(p_{23}^*(t) + p_{24}^*(t)\right)z^{7.9}. \\ &+ p_{15}(t)p_{21}^*(t)z^{0} + p_{15}(t)p_{22}^*(t)z^{5.2} + p_{15}(t)\left(p_{23}^*(t) + p_{24}^*(t)\right)z^{10.5}. \end{aligned} \quad (29)$$

In the resulting UGF *U*(*z*,*t*) the powers of z are found as minimum of powers of corresponding terms.

Taking into account that $p_{11}(t) + p_{12}(t) + p_{13}(t) + p_{14}(t) + p_{15}(t) = 1$ and $p_{21}^*(t) + p_{22}^*(t) + p_{23}^*(t) + p_{24}^*(t) = 1$, one can simplify the last expression for *U*(*z*,*t*) and obtain the resulting UGF associated with the output performance stochastic process **g**, **p**(*t*) of the entire MSS in the following form

$$U(z,t) = \sum_{i=1}^{5} p_i(t) z^{g_i} \quad (30)$$

where

$$\begin{aligned} g_1 &= 0, & p_1(t) &= p_{11}(t) + \left[1 - p_{11}(t)\right]p_{21}^*(t), \\ g_2 &= 2.6 \cdot 10^9 \text{ BTU/year}, & p_2(t) &= p_{12}(t)\left[1 - p_{21}^*(t)\right], \\ g_3 &= 5.2 \cdot 10^9 \text{ BTU/year}, & p_3(t) &= p_{13}(t)\left[1 - p_{21}^*(t)\right] + \left[p_{14}(t) + p_{15}(t)\right]p_{22}^*(t), \\ g_4 &= 7.9 \cdot 10^9 \text{ BTU/year}, & p_4(t) &= p_{14}(t)\left[p_{23}^*(t) + p_{24}^*(t)\right], \\ g_5 &= 10.5 \cdot 10^9 \text{ BTU/year}, & p_5(t) &= p_{15}(t)\left[p_{23}^*(t) + p_{24}^*(t)\right]. \end{aligned}$$

These two sets

$\mathbf{g} = \{g_1, g_2, g_3, g_4, g_5\}$ and $\mathbf{p}(t) = \{p_1(t), p_2(t), p_3(t), p_4(t), p_5(t)\}$

completely define output performance stochastic process for the entire MSS.

Based on resulting UGF $U(z,t)$ of the entire MSS, one can obtain the MSS reliability indices. The instantaneous MSS availability for the constant demand level $w=5.0\cdot10^9$ BTU per year

$$A(t) = \delta_A(U(z,t),w) = \delta_A(\sum_{i=1}^{5} p_i(t)z^{g_i},5) = \sum_{i=1}^{5} p_i(t)1(F(g_i,5)\geq 0) = p_3(t)+p_4(t)+p_5(t). \quad (31)$$

The instantaneous mean output performance at any instant $t>0$

$$E(t) = \delta_E(U(z,t)) = \sum_{i=1}^{5} p_i(t)g_i = 2.6p_2(t)+5.2p_3(t)+7.9p_4(t)+10.5p_5(t). \quad (32)$$

The instantaneous performance deficiency $D(t)$ at any time t for the constant demand $w=5.0\cdot10^9$ BTU per year:

$$D(t) = \delta_D(U(z),w) = \sum_{i=1}^{5} p_i(t)\cdot\max(5-g_i,0) = = p_1(t)(5-0)+p_2(t)(5-2.6) = 5p_1(t)+2.4p_2(t). \quad (33)$$

Calculated reliability indices $A(t)$, $E(t)$ and $D(t)$ are presented on the Figures 6-8.

Note that instead of solving the system of $K=5*4=20$ differential equations (as it should be done in the straightforward Markov method) here we solve just two systems. The further derivation of the entire system states probabilities and reliability indices is based on using simple algebraic equations.

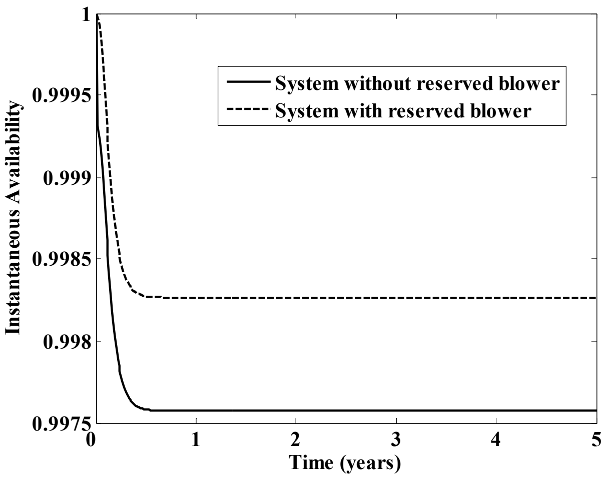

Figure 6. MSS instantaneous availability for different types of systems

Curves in Figures 6-8 support the engineering decision-making and determine the areas where required performance deficiency level of the refrigeration system can be provided by configuration "with additional blower" or by configuration "without additional blower". For example, from the Figure 6 one can conclude that the configuration "without additional blower" cannot provide the required average availability, if it is greater than 0.998.

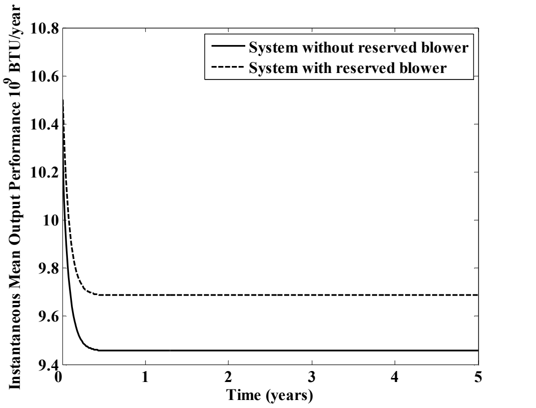

Figure 7. MSS instantaneous mean output performance for different types of systems

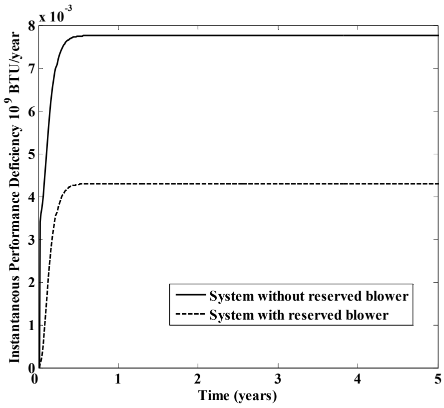

Figure 8. MSS instantaneous mean performance deficiency for different types of systems

4. Conclusions

The universal method was applied to compute MSS reliability measures: system availability, output performance and performance deficiency. The method is based on the combined Universal Generating Functions and stochastic processes method.

The case-study demonstrates that the approach is well formalized and suitable for practical application in reliability engineering. It supports the engineering decision-making and determines different system structures providing a required reliability/availability level of MSS.

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