GENERALIZED BURR-HATKE EQUATION AS GENERATOR OF A HOMOGRAPHIC FAILURE RATE

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Abstract: *Starting from a generalized form of Burr-Hatke differential equation namely* $\mathrm{d}\varphi / \mathrm{d}x = u(x) \cdot \varphi^a + v(x) \cdot \varphi^b$ *(see for details [18]), we obtain – for peculiar choices of u, v, a and b – as a solution, a reliability function* $\varphi(x)$ *which provides a homographic failure rate (HFR)* $h(x) = \theta + 1/x$, *where* $\theta > 0$. *Some statistical inferences on the variable X having such a hazard rate are performed. Main indicators are evaluated and it is proved that the maximum likelihood method of estimation cannot furnish a solution for the parameter involved. A moment estimator is deduced which is used in the construction of some special sampling plans for durability testing.*

Key words: *Burr-Hatke equation; HFR –homographic failure rate; exponential integral; moment estimator; log-likelihood equation; (n, 0) - sampling plan*

1. Preliminaries and some historical remarks

In statistical distribution theory, the so-called Burr-Hatke differential equation, namely

$$\frac{\mathrm{d}F}{\mathrm{d}x} = F\left(1-F\right)g\left(x,F\right) \quad \text{with } F_0 = F\left(x_0\right),\ x_0 \in R \tag{1.1.}$$

where $F(x) = \mathrm{Prob}\left\{X \le x\right\}$ is the c.d.f. (cumulative distribution function) of a continuous random variable X and $g\left(x,F\right)$ is an arbitrary positive function for any $x \in R$ – is considered by many authors as a system of c.d.f.(s) generator or simply a **system of frequency curves** (see Rodriguez [16[1], page 218] or Johnson et al. [13, page 54]).

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Burr [4] and (Sister) Hatke [11] presented a list of twelve such cdf(s) – denoted from I to XII, the most famous being probably the form II, which is just the "logistic curve" introduced by the Belgian scientist François Pierre VERHULST (1804-1849) – see *Iosifescu et all* [12, page 285]).

The majority of cdf(s) are deduced from (1.1.) by taking $g\left(x,F\right)=g\left(x\right)$ – that is a simpler choice. The form XII – for which Rodriguez elaborated even a "guide" (see [15]) has received more attention than the other ones in this family.

The form XII, that is $F(x;c,k)=1-\left(1+x^{c}\right)^{-k},\quad x\geq 0,\ c,k>0$ has been used in sampling inspection theory (Zimmer and Burr [20]). Burr also showed that for $c\approx 4.874$ and $k\approx 6.158$ we find that $E(X)\approx 0.6447$ and $\sqrt{Var(X)}\approx 0.1620$ (where *X* is the Burr-type XII variable) and the normed variable $U=\left[X-E\left(X\right)\right]/\sqrt{Var(X)}$ approximates quite well the standardized normal variable *N* (0,1) – see further developments on this subject matter in Burr [6], Burr and Cislak [7] and Burr [8].

In the last two decades, the Burr-type X distribution, that is

$$F\left(x;a,b\right)=\left[1-\exp\left(-ax^{2}\right)\right]^{b},\quad x\geq 0,\quad a,b>0 \tag{1.2.}$$

has attracted the interest of reliability engineers as well as of statisticians since if "can be used quite effectively in modeling strength data and also modeling general lifetime data" as Raqab and Kundu [14] claimed.

In fact, (1.2/) is a generalization of Rayleigh cdf: for *b* = 1we obtain the classical form proposed in 1880 by Lord Rayleigh (1842-1919), as the distribution of the amplitude resulting from the harmonic oscillations. Since $F(x;a,1)$ provides a linearly increasing failure (hazard) rate $h(x)=F'/\left(1-F\right)=2ax,\quad x\geq 0,a>0$ this peculiar Burr-type X distribution is suitable to describe the irreversible wear-out processes which take place in metalworking (grinding and cutting-tool durability analysis).

The Burr-type XII distribution has been used as a failure model especially in the case of censored and multicensored / progressively censored data (Wingo [19]).

The form (1.1.) has been generalized (see the second author [18]) as below

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=u(x)\varphi^{a}+v(x)\varphi^{b} \tag{1.3.}$$

where $\varphi(x)$ is a positive function for every $x\geq 0,\ u(x)$ and $v(x)$ are continuous functions and *a* and *b* are two arbitrary real numbers. If φ is a cdf and $u(x)=1,\ v(x)=-1,\ a=1$ and $b=2$, **one recovers the Burr-Hatke form (1.1).**

The above proposal (1.3.) has the advantage that it can provide not only cdf(s) – as (1.1) does – but also pdf(s) – probability density functions and reliability ones. In other words, it is more flexible.

For instance, if we take $u(x)=v(x)=kx^{k-1}/\theta,\ x\geq 0,\ \theta,k>0, a=0$ and $b=1$, we shall get $\varphi(x)=1-\exp\left(-x^{k}/\theta\right)$ which is the famous cdf proposed in 1951 by the Swedish scientist Waloddi WEIBULL (1887-1979) – see Johnson et al. [13, page 628].

If we choose $u(x)=(x/2\pi)\exp\left(-x^2\right), x\in R,\ v(x)=0, a=-1$, for any $b\in R$ we shall obtain the standardized normal pdf $\varphi(x)=\left(1/\sqrt{2\pi}\right)\exp\left(-x^2/2\right)$.

Let us take now a = b = 1, u(x) = -1/x, x ≥ 1, v(x) = - θ, with θ > 0 and consequently, from (1.3.) we shall get

$$\varphi(x)=\frac{1}{x}\exp\left[-\theta\left(x-1\right)\right],\quad x\geq 1, \theta>0 \tag{1.4.}$$

which is a **reliability function** ($\varphi(1)=1$, $\varphi(\infty)=0$).

The purpose of this paper is to perform a statistical analysis on the random variable having (1.4.) as its survivor function.

2. Straightforward consequences

From (1.4) we can deduce immediately the cdf and pdf of the underlying variable.

$$F(x;\theta)=1-x^{-1}\cdot\exp\left[-\theta\left(x-1\right)\right],\quad x\geq 1, \theta>0 \tag{2.1.}$$

$$f(x;\theta)=\left(\theta x^{-1}+x^{-2}\right)\cdot\exp\left[-\theta\left(x-1\right)\right],\quad x\geq 1,\ \theta>0. \tag{2.2.}$$

Since $f'(x;\theta)$ is always strictly negative, $f(x;\theta)$ is strictly decreasing: the curve associated to *f* has the starting point of coordinates $\left(1, 1+\theta\right)$ and a horizontal asymptote $y=0$.

The density curve is therefore of an exponential type which decreases faster than the classical exponential pdf.

The associated hazard rate is:

$$h\ (x;\theta)=\frac{f(x;\theta)}{1-F(x;\theta)}=\theta'+\frac{1}{x}=\frac{\theta x+1}{x},\quad x\geq 1, \theta>0 \tag{2.3.}$$

with $h'\left(x;\theta\right)=-1/x^2<0$, that is *h* is decreasing and in fact it is a peculiar form of the general homographic function. Therefore, we investigate a HFR-type variable.

Another form of a homographic hazard function, namely $h_1\left(x;\theta\right)=\theta^2 x/\left(1+\theta x\right)$, $x\geq 0, \theta>0$ has been studied by Bârsan-Pipu et al. [2] but this is strictly increasing since $h_1'=\theta^2/\left(1+\theta x\right)^2>0$.

Therefore our form (2.3.) is adequate for modeling the "burn-in" process (or "infant mortality" in demographic terms), meanwhile $h_1(x;\theta)$ is suitable for fatigue and wear-out cases.

3. Main indicators and parameter estimation

First, we shall compute the first four raw moments of our HFR-variable – let it be *X*. To this purpose, we shall state

Property 1. The non-central m^{th} moment of X , with $m\geq 2$ can be expressed explicitly as

$$E\left(X^m\right)=1+\frac{m}{\theta}\cdot\sum_{j=0}^{m-2}\frac{(m-2)!}{(m-j-2)!}\theta^{-j},\quad m=2,3,... \tag{3.1.}$$

The mean-value (the first raw moment) implies a special function – namely the exponential integral with negative argument, $Ei(-u)$.

$$E\left(X^2\right)=\int_1^\infty x^2 f(x;\theta)\,\mathrm{d}x=\int_1^\infty e^{-\theta(x-1)}\mathrm{d}x+\theta\int_1^\infty xe^{-\theta(x-1)}\mathrm{d}x \tag{3.2.}$$

Proof. We shall reach (3.1.) sequentially. For *m* = 2 we have

$$\theta\left[-\frac{x}{\theta}e^{-\theta(x-1)}\Big|_1^\infty+\frac{1}{\theta}\int_1^\infty e^{-\theta(x-1)}\mathrm{d}x\right]=\frac{1}{\theta}+1+\frac{1}{\theta}=1+2/\theta.$$

The expression is the same if we take *m* = 2 in (3.1.).

After some more similar integration by parts we shall get

$$E\left(X^3\right)=1+3(1+1/\theta)/\theta \tag{3.3.}$$

$$E\left(X^4\right)=1+4\left(1+2/\theta+2/\theta^2\right)/\theta \tag{3.4.}$$

For *m* = 3 and *m* = 4 in (3.1.) one obtains easily (3.2.) and (3.3.). A simple induction will validate (3.1.).

For *m* = 1, we have

$$E(X)=\int_1^\infty xf(x;\theta)\,\mathrm{d}x=\int_1^\infty\left(\theta+\frac{1}{\mathrm{x}}\right)e^{-\theta(x-1)}\mathrm{d}x=\theta\int_1^\infty e^{-\theta(x-1)}\mathrm{d}x+e^{\theta}\int_1^\infty\frac{e^{-\theta x}}{x}\mathrm{d}x=$$

$$=1-e^{\theta}\cdot Ei(-\theta) \tag{3.5.}$$

– see Abramowitz-Stegun [1, page 56] or Smoleanski [17, page 121] where the indefinite integral generating this special function is provided as

$$\int\frac{e^{-ax}}{x}\mathrm{d}x=\ln|x|+\frac{ax}{1!}+\frac{a^2x^2}{2\cdot 2!}+\cdots+\frac{a^nx^n}{n\cdot n!}+\cdots \tag{3.6.}$$

(here $a\neq 0$ could be positive or negative).

The straightforward consequence is following: one cannot use $E(X)$ in order to estimate θ by applying method of moments proposed in 1891 by Karl PEARSON (1857-1936). Nevertheless we may state

Property 2. The moment estimator for θ has the below form

$$\hat{\theta}_M=\frac{2}{n^{-1}\sum x_i^2-1} \tag{3.7.}$$

where $x_i,\ i=\overline{1,n}$ are sample values on X $\left(x_i\geq 1,\forall i\right)$.

The proof is immediate: we have to equate $E\left(X^2\right)$ with the empirical raw second order moment $n^{-1}\sum x_i^2$.

Property 3. The log-likelihood equation associated to $f(x;\theta)$ has all its roots negative ones, therefore there is no MLE – Maximum Likelihood Estimator for θ.

Proof. We may write successively

$$L=\prod_{i=1}^{n} f\left(x_i;\theta\right)=\left[\prod_{i=1}^{n}\left(x_i^{-2}+\theta\cdot x_i^{-1}\right)\right]e^{-n\theta}\cdot\exp\left(-\theta\sum_{i=1}^{n}x_i\right) \quad (3.8.)$$

$$\ln L=\prod_{i=1}^{n}\ln\left(x_i^{-2}+\theta\cdot x_i^{-1}\right)-\theta\sum_{i=1}^{n}x_i+n\theta \quad (3.9.)$$

$$\frac{\partial \ln L}{\partial \theta}=\sum_{i=1}^{n}\frac{x_i}{1+\hat{\theta}x_i}-\sum_{i=1}^{n}x_i+n=0 \quad (3.10.)$$

If we denote $u_i=1/x_i$ we shall have

$$\sum_{i=1}^{n}\frac{1}{\hat{\theta}+u_i}+n-\sum_{i=1}^{n}\frac{1}{u_i}=0 \quad (3.11.)$$

Since $1/x_i<1$, the quantity $\left(n-\sum_{i=1}^{n}\frac{1}{u_i}\right)$ is always positive and hence the equation (3.11.) has real roots but all negative ones. Indeed, the function

$$\psi\left(\hat{\theta}\right)=\sum_{1}^{n}\frac{1}{\hat{\theta}+u_i}+n-\sum_{1}^{n}\frac{1}{u_i} \quad (3.12.)$$

is a decreasing one, since $\psi'\left(\theta\right)<0$. It has a horizontal asymptote

$$\hat{\theta}=n-\sum_{i=1}^{n}\frac{1}{u_i}\ \in\left[0,n\right) \quad (3.13.)$$

For $\hat{\theta}=0$, we have $\psi\left(0\right)=n$, that is the point (0, *n*) is on the vertical coordinate axis. There exists also *n* vertical asymptotes, namely $\hat{\theta}=-u_i<0,\quad i=\overline{1,n}$ since $\lim_{\hat{\theta}\to u_i}\psi\left(\hat{\theta}\right)=\pm\infty$ and therefore there are *n* “cuts” on the horizontal axis that is we have *n real negative roots.*

It follows that the maximum likelihood method (MLM) does not provide an estimator for θ.

Remark: this situation is not unique; for the so-called power law $F\left(x;\delta,b\right)=\left(x/b\right)^{\delta},\quad \delta>0,\ 0\le x\le b$, the MLM cannot give MLE(s) for δ and *b* (such an estimator exists only if *b* is assumed to be known) – see Bârsan-Pipu et al. [3, pages 73-75].

4. Special sampling plans

In this section we shall construct a sampling plan of (*n*, 0) type for durability testing in the case that the time-to-failure obeys the law $F\left(x;\theta\right)$ given by (2.1.).

The problem is in this case to establish the sample size *n* when the acceptance number is A = 0. We have a lot of size *N* with a given fraction defective *p* and we fix a testing time T_0 for a sample of *n* product units submitted to a durability test. We have hence (see Derman and Ross [10])

$$p = F(T_0;\theta) = 1 - \frac{1}{T_0} \cdot \exp\left[-\theta(T_0 - 1)\right] \quad (4.1.)$$

$$1 - p = T_0^{-1} \cdot \exp\left[-\theta(T_0 - 1)\right]. \quad (4.2.)$$

Then, the probability (β) to accept the batch during the T_0 testing period, via *n* tested elements is

$$\beta = (1-p)^n = T_0^{-n} \cdot \exp\left[-n \cdot \theta(T_0 - 1)\right] \quad (4.3.)$$

and by taking the logarithm we have

$$\ln \beta = -n\theta(T_0 - 1) - n \ln T_0 \quad (4.4.)$$

which gives

$$n = \left[\frac{-\ln \beta}{(T_0 - 1)\theta + \ln T_0}\right], \ [m] - \text{the nearest integer to } m. \quad (4.5.)$$

In (4.5.) θ has to be replaced by its moment estimator given by (3.7.), where $E(X^2)$ – that is the mean-durability of square failure times represented by X^2 has a previous specified value.

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