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THE WEIGHTED POSSIBILISTIC MEAN VARIANCE AND COVARIANCE OF FUZZY NUMBERS

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Abstract: *In the context of Markowitz portfolio selection problem (Markowitz,1959), this paper develops a "weighted" possibilistic mean-variance and covariance portfolio selection model. In the process of selecting a portfolio, this work is concerned with the choice of the portfolio (Markowitz,1952).*

Key words: Portfolio selection; VaR; Possibilistic mean value; Possibilistic theory; Weighted possibilistic mean; Fuzzy theory

1. Introduction

This article is organized in the following way: Section 2 propose an overview of proportional transaction costs model; in Section 3 we consider the possibilistic theory proposed by Zadeh [15][2] and we present the rate of return on security given by a trapezoidal fuzzy number; in Section 4, we stated the weighted possibilistic mean variance and covariance of fuzzy numbers. Thus are extended some recently results in this field [8,12,15].

2. The proportional transaction costs model

Transaction cost is one of the main sources of concern to managers see [1, 16].

Assume the rate of transaction cost of security *j* (*j*=1,.., *n*) and allocation of *i* (*i* =1,…, *k*) assets is c_{ji}, thus the transaction cost of security *j* and allocation of *i* assets is $c_{ji}x_j$. The transaction cost of portfolio $x=(x_1,...,x_n)$ is $\sum_{j=1}^{n} c_{ji}x_j,\ i=1,...,k$. Considering the proportional transaction cost and the shortfall probability constraints, we purpose the following mean *VaR* portfolio selection model with transaction costs [14]:

$$\underset{x\in R^n}{Max}\left[E(v_1)-\sum_{j=1}^{n}c_{j1}x_j,\ ....,\ E(v_k)-\sum_{j=1}^{n}c_{jk}x_j\right] \tag{2.1}$$

Subject to $\Pr\{v_i<(VaR)_i\}\le\beta_i,\ i=1,...,k,$ (2.2)

$$\sum_{j=1}^{n} x_j = 1, \tag{2.3}$$

$$M_{1j} \le x_j \le M_{2j}, \quad j = 1,...,n. \tag{2.4}$$

3. Triangular and trapezoidal fuzzy numbers

We consider the possibilistic theory proposed by Zadeh [15]. Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers with membership functions $\mu_{\widetilde{a}}$ and $\mu_{\widetilde{b}}$ respectively. The possibility operator (*Pos*) is defined in [12].

Let the rate of return on security given by a trapezoidal fuzzy number $\widetilde{r} = (r_1, r_2, r_3, r_4)$ where $r_1 < r_2 \le r_3 < r_4$. Then the membership function of the fuzzy number $\widetilde{r}$ can be denoted by:

$$\mu(x) = \begin{cases} \dfrac{x - r_1}{r_2 - r_1} & , r_1 \le x \le r_2, \\ 1 & , r_2 \le x \le r_3, \\ \dfrac{x - r_4}{r_3 - r_4} & , r_3 \le x \le r_4, \\ 0 & , otherwise \end{cases} \tag{3.1}$$

We mention that trapezoidal fuzzy number is *triangular* fuzzy number if $r_2 = r_3$.

Let us consider two trapezoidal fuzzy numbers $\widetilde{r} = (r_1, r_2, r_3, r_4)$ and $\widetilde{b} = (b_1, b_2, b_3, b_4)$.

If $r_2 \le b_3$, then we have

$$Pos(\widetilde{r} \le \widetilde{b}) = \sup\{ \min\{\mu_{\widetilde{r}}(x), \mu_{\widetilde{b}}(y)\} | \, x \le y\}$$
$$\ge \min\{ \mu_{\widetilde{r}}(r_2), \mu_{\widetilde{b}}(b_3) \} = \min\{1,1\} = 1,$$

which implies that $Pos(\widetilde{r} \le \widetilde{b}) = 1$. If $r_2 \ge b_3$ and $r_1 \le b_4$ then the supremum is achieved at point of intersection δ_x of the two membership function $\mu_{\widetilde{r}}(x)$ and $\mu_{\widetilde{b}}(x)$. A simple computation shows that

$$Pos(\widetilde{r} \le \widetilde{b}) = \delta = \frac{b_4 - r_1}{(b_4 - b_3) + (r_2 - r_1)}$$

and

$$\delta_x = r_1 + (r_2 - r_1)\delta$$

If $r_1 > b_4$, then for any $x < y$, at least one of the equalities $\mu_{\widetilde{r}}(x) = 0, \ \mu_{\widetilde{b}}(y) = 0$ hold. Thus we have $Pos(\widetilde{r} \le \widetilde{b}) = 0$. Now we summarize the above results as

$$Pos(\widetilde{r} \le \widetilde{b}) = \begin{cases} 1, & r_2 \le b_3 \\ \delta, & r_2 \ge b_3, r_1 \le b_4 \\ 0, & r_1 \ge b_4 \end{cases} \tag{3.2}$$

Especially, when $\widetilde{b}$ is the crisp number 0, then we have

$$Pos\left(\widetilde{r} \le 0\right) = \begin{cases} 1, & r_2 \le 0 \\ \delta, & r_1 \le 0 \le r_2 \\ 0, & r_1 \ge 0 \end{cases} \tag{3.3}$$

where

$$\delta = \frac{r_1}{r_1 - r_2}. \tag{3.4}$$

We now turn our attention the following lemma.

Lemma 3.1 [4] *Assume that trapezoidal fuzzy number* $\widetilde{r} = (r_1, r_2, r_3, r_4)$. *Then for any given confidence level* α *with* $0 \le \alpha \le 1$, $Pos(\widetilde{r} \le 0) \ge \alpha$ *if and only if* $(1-\alpha)r_1 + \alpha r_2 \le 0$.

The λ level set of a fuzzy number $\widetilde{r} = (r_1, r_2, r_3, r_4)$ is a crisp subset of R and denoted by $[\widetilde{r}]^\lambda = \{x | \mu(x) \ge \lambda, x \in R\}$, then according to Carlsson et al [3], we have

$$[\widetilde{r}]^\lambda = \{x | \mu(x) \ge \lambda, x \in R\} = [r_1 + \lambda(r_2 - r_1), r_4 - \lambda(r_4 - r_3)].$$

Given $[\widetilde{r}]^\lambda = [a_1(\lambda), a_2(\lambda)]$, the crisp possibilistic mean value of $\widetilde{r} = (r_1, r_2, r_3, r_4)$ is

$$\widetilde{E}(\widetilde{r}) = \int_0^1 \lambda(a_1(\lambda) + a_2(\lambda))d\lambda,$$

where $\widetilde{E}$ *denotes fuzzy mean operator*.

We can see that if $\widetilde{r} = (r_1, r_2, r_3, r_4)$ is a trapezoidal fuzzy number then

$$\widetilde{E}(\widetilde{r}) = \int_0^1 \lambda(r_1 + \lambda(r_2 - r_1) + r_4 - \lambda(r_4 - r_3))d\lambda = \frac{r_2 + r_3}{3} + \frac{r_1 + r_4}{6}. \tag{3.5}$$

4. The weighted possibilistic mean variance and covariance of fuzzy numbers

The classical mean-variance portfolio selection problem uses the variance as the measure for risk, which puts the same weight on the down side and upside of the return. In this section, we study the "weighted" possibilistic mean-variance and covariance portfolio selection model.

Definition 4.2 [6]) *Let* $\widetilde{r} \in \mathrm{F}$ *be a fuzzy number with* $[\widetilde{r}]^\lambda = [r_1(\lambda), r_2(\lambda)]$, $\lambda \in [0,1]$. *The* w*-weighted possibilistic variance of* $\widetilde{r}$ *is*

$$Var_w(\widetilde{r}) = \int_0^1 \left(\frac{r_2(\lambda) - r_1(\lambda)}{2}\right)^2 w(\lambda)d\lambda$$

$$= \int_0^1 \frac{1}{2}\left(\left[\frac{r_1(\lambda) + r_2(\lambda)}{2} - r_1(\lambda)\right]^2 + \left[r_2(\lambda) - \frac{r_1(\lambda) + r_2(\lambda)}{2}\right]^2\right) w(\lambda)\, d\lambda$$

where weighting function is non-decreasing and satisfies

$$\int_0^1 w(\lambda)d\lambda = 1. \quad (4.1)$$

The standard deviation of $\widetilde{r}$ is defined by

$$\sigma_{\widetilde{r}} = \sqrt{Var(\widetilde{r})} \quad (4.2)$$

Let $\widetilde{r}$ fuzzy number and w be a weighting function, we define the weighted possibilistic variance of $\widetilde{r}$ by

$$Var_w(\widetilde{r}) = \int_0^1 \left(\frac{r_2(\lambda) - r_1(\lambda)}{2} \right)^2 w(\lambda)d\lambda$$

and the weighted covariance of $\widetilde{r}$ and $\widetilde{b}$ is defined as

$$Cov_w(\widetilde{r}, b) = \int_0^1 \left(\frac{r_2(\lambda) - r_1(\lambda)}{2} . \frac{b_2(\lambda) - b_1(\lambda)}{2} \right) w(\lambda)d\lambda \quad .$$

If $w(\lambda) = 2\lambda,\ \lambda \in [0,1]$

$$Var_w(\widetilde{r}) = \frac{1}{2}\int_0^1 (r_2(\lambda) - r_1(\lambda))\lambda d\lambda\ , \quad (4.3)$$

and

$$Cov_w(\widetilde{r}, b) = \frac{1}{2}\int_0^1 (r_2(\lambda) - r_1(\lambda)(b_2(\lambda) - b_1(\lambda))2\lambda d\lambda\ . \quad (4.4)$$

Let $\widetilde{r} = (r_1, r_2, r_3, r_4)$ and $\widetilde{b} = (b_1, b_2, b_3, b_4)$ be fuzzy numbers of trapezoidal form.

Let $w(\lambda) = (2\gamma - 1)\left((1-\lambda)^{-\frac{1}{2\gamma}} - 1 \right)$,

where $\gamma \ge 1$, be a weighting function then the power-weighted variance and covariance $\widetilde{r}$ and $\widetilde{b}$ are computed by

$$Var_w(\widetilde{r}) = (2\gamma - 1)\int_0^1 \left(\frac{r_2(\lambda) - r_1(\lambda)}{2} \right)^2 \left((1-\lambda)^{-\frac{1}{2\gamma}} - 1 \right) d\lambda$$

$$= \frac{(2\gamma-1)}{4}\left[\frac{(r_2 - r_1)^2}{2\gamma - 1} + 2(r_2 - r_1)(\alpha + \beta)\frac{2(r_2 - r_1)(r_3 + r_4)}{2(4\gamma - 1)} + (\alpha+\beta)^2 \frac{(r_3 + r_4)^2}{3(6\gamma - 1)} \right] \quad (4.5)$$

$$Cov_w(\widetilde{r}, \widetilde{b}) = (2\gamma - 1)\int_0^1 \frac{(r_2 - r_1)((1-\lambda)(r_3 + r_4)}{2} . \frac{(b_2 - b_1) + (1-\lambda)(b_3 + b_4)}{2}\left((1-\lambda)^{-\frac{1}{2\gamma}} - 1 \right) d\lambda \quad (4.6)$$

$$= \frac{(2\gamma-1)}{4}\left[\frac{r_2 - r_1)(b_2 - b_1)}{2\gamma - 1} + \frac{(r_2 - r_1)(b_3 + b_4)(r_3 + r_4)}{2(4\gamma - 1)} + \frac{(r_3 + r_4)(b_3 + b_4)}{3(6\gamma - 1)} \right].$$

Theorem 4.1 [12] *The mean-variance efficient portfolio model is*

$$\max_{x \in R^n} \sum_{i=1}^{q} \lambda_i \left[\overline{\widetilde{E}}_w \left(\sum_{j=1}^{n} \widetilde{r}_{ji} x_j \right) - \sum_{j=1}^{n} c_{ji} x_j \right] \quad (4.7)$$

subject to $Pos\left(\sum_{j=1}^{n} \widetilde{r}_{ji} x_j < \widetilde{b}_i \right) \le \beta_i ,\quad i = \overline{1, q}\ ,$ (4.8)

$$\sum_{j=1}^{n} x_j = 1, \tag{4.9}$$

$$M_{1j} \le x_j \le M_{2j}, \quad j = \overline{1,n}. \tag{4.10}$$

In the next theorem we extend ([12], Theorem 4.2) to the case weighted possibility mean- variance approach with a special weighted $w(\lambda)$.

Theorem 4.2 *Let* $w(\lambda) = (2\gamma - 1)\left[(1-\lambda)^{-\frac{1}{2\gamma}} - 1\right]$, $\gamma \ge 1$ *the weighted possibility mean variance (covariance) of the trapezoidal number* $\widetilde{r}_{ji} = \left(r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4}\right)$ *where* $r_{(ji)1} < r_{(ji)2} \le r_{(ji)3} < r_{(ji)4}$ *and addition* $\widetilde{b}_i = \left(b_{i1}, b_{i2}, b_{i3}, b_{i4}\right)$ *is a trapezoidal number for (VaR)*$_i$ *level,* $i = \overline{1,q}$. *For* $\lambda_i > 0, i = \overline{1,q}$, *then the possibilistic mean variance (covariance) portfolio selection model is*

$$\max_{x \in R^n} \sum_{i=1}^{q} \lambda_i \left[\frac{(2\gamma-1)\left(\sum_{j=1}^{n} r_{(ji)2} x_j - \sum_{j=1}^{n} r_{(ji)1} x_j\right)^2}{4(2\gamma-1)} + \frac{(2\gamma-1)\left(\sum_{j=1}^{n} r_{(ji)2} x_j - \sum_{j=1}^{n} r_{(ji)1} x_j\right)\left(\sum_{j=1}^{n} r_{(ji)3} x_j + \sum_{j=1}^{n} r_{(ji)4} x_j\right)}{4(4\gamma-1)} \right.$$

$$\left. + \frac{(2\gamma-1)\left(\sum_{j=1}^{n} r_{(ji)3} x_j + \sum_{j=1}^{n} r_{(ji)4} x_j\right)^2}{12(6\gamma-1)} - \sum_{j=1}^{n} c_{ji} x_j \right] \tag{4.11}$$

subject to

$$(1-\beta_i)\left(\sum_{j=1}^{n} r_{(ji)1} x_j - b_{i4}\right) + \beta_i \left(\sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3}\right) \ge 0, i = \overline{1,q}, \tag{4.12}$$

$$\sum_{j=1}^{n} x_j = 1, \tag{4.13}$$

$$M_{1j} \le x_j \le M_{2j}, \quad j = \overline{1,n}. \tag{4.14}$$

Proof : From the equation (3.5), we have

$$\widetilde{\widetilde{E}}_w\left(\sum_{j=1}^{n}\widetilde{r}_{ji}x_j\right)=\left[\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)2}x_j-\sum_{j=1}^{n}r_{(ji)1}x_j\right)^2}{4(2\gamma-1)}+\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)2}x_j-\sum_{j=1}^{n}r_{(ji)1}x_j\right)\left(\sum_{j=1}^{n}r_{(ji)3}x_j+\sum_{j=1}^{n}r_{(ji)4}x_j\right)}{4(4\gamma-1)}\right.$$

$$\left.+\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)3}x_j+\sum_{j=1}^{n}r_{(ji)4}x_j\right)^2}{12(6\gamma-1)}\right],\ i=\overline{1,q},\ \gamma\geq 1.$$

From Lemma 3.1, we have that

$Pos\left(\sum_{j=1}^{n}\widetilde{r}_{ji}x_j<\widetilde{b}_i\right)\leq\beta_i$, $i=\overline{1,q}$ is equivalent with

$$(1-\beta_i)\left(\sum_{j=1}^{n}r_{(ji)1}x_j-b_{i4}\right)+\beta_i\left(\sum_{j=1}^{n}r_{(ji)2}x_j-b_{i3}\right)\geq 0,\ i=\overline{1,q}.$$

Furthermore, from (4.11)-(4.14) given by Theorem 4.1, we get the following form :

$$\max_{x\in R^n}\sum_{i=1}^{q}\lambda_i\left[\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)2}x_j-\sum_{j=1}^{n}r_{(ji)1}x_j\right)^2}{4(2\gamma-1)}+\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)2}x_j-\sum_{j=1}^{n}r_{(ji)1}x_j\right)\left(\sum_{j=1}^{n}r_{(ji)3}x_j+\sum_{j=1}^{n}r_{(ji)4}x_j\right)}{4(4\gamma-1)}\right.$$

$$\left.+\frac{(2\gamma-1)\left(\sum_{j=1}^{n}r_{(ji)3}x_j+\sum_{j=1}^{n}r_{(ji)4}x_j\right)^2}{12(6\gamma-1)}-\sum_{j=1}^{n}c_{ji}x_j\right]\tag{4.15}$$

subject to

$$(1-\beta_i)\left(\sum_{j=1}^{n}r_{(ji)1}x_j-b_{i4}\right)+\beta_i\left(\sum_{j=1}^{n}r_{(ji)2}x_j-b_{i3}\right)\geq 0,\ i=\overline{1,q}\tag{4.16}$$

$$\sum_{j=1}^{n}x_j=1,\tag{4.17}$$

$$M_{1j}\leq x_j\leq M_{2j},\quad j=\overline{1,n}.\tag{4.18}$$

This completes the proof. □

Problem (4.15)-(4.18) is a standard multi-objective linear programming problem. Also we can obtain an optimal solution by using some algorithms of multi-objective programming [3, 11].

For $\gamma \to \infty$ we see that $\lim_{\gamma \to \infty} \overline{\overline{E}}_w(\widetilde{r}) = \frac{(r_2 - r_1)^2}{4} + \frac{(r_2 - r_1)(r_3 + r_4)}{8} + \frac{(r_3 + r_4)^2}{36}$.

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