STRENGTH OF FACTORS IN 3^3 FACTORIAL DESIGNS USING BAYESIAN ANALYSIS

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Abstract
The study proposes to consider factorial design at three levels and identify all significant factors based on its inherent strength. The methodology considers full, fractional, and reduced factorial designs with three factors each at three levels, to examine the effectiveness of factors in these models through simulation and employing real data. By identifying and quantifying the Bayes factors through simulated datasets, the true strength of the main/interaction effects in these three designs were discovered. Finally, the study concludes that reduced factorial design produces better results than traditional one-third fractional factorial designs when there are no other constraints to adding more factors to the model for analysis.

Keywords: 3^3*factorial design; Zellner's g prior; Jeffreys-Zellner-Siow prior; Hyper- g priors; strength of factors*

Introduction

Factorial designs are being widely used in experiments involving several factors and where it is necessary to study the impact of the factors or combination of factors on a process. The main goal is to identify the significant factors from a set of main effects and interactions. It may be noted that as the number of factors increases the total number of combinations becomes unwieldy. The present research considered the factorial design with p factors, each at three levels such as "low", "intermediate" and "high" levels of a factor. The factorial designs at three levels have been well exploited and analyzed considering all factors, confounded completely and partially and lastly as fractional factorials.Xu et al. (2009) discussed developments in non-regular fractional factorial designs, particularly optimality criteria, projection properties, resolutions, and aberration criteria. Baba et al. (2013) proposed the usefulness of the empirical Bayesian approach to the saturated factorial designs and observed predictions and inferences for the parameters. Espinosa et al. (2016) proposed a new approach to screen for active factorial effects from replicated factorial design using the potential outcomes framework and based on sequential posterior predictive model checks. Rouder et al. (2017) presented the Bayes Factor approach to multi-way ANOVA with hierarchical models for fixed, random effects, with-subjects, between-subjects, and mixed designs. Schwaferts and Augustin (2018) applied Bayes Factors to get optimal decisions in a study on the framework of hypothesis-based Bayesian decision theory with robust loss function and step-by-step guidelines. Khaw et al. (2019) identified the best extraction condition of factors from the application of the six-factor fractional factorial design. Lakens et al. (2020) have provided comprehensive explanations of the calculation and interpretation of Bayes Factors for several tests. In educational research, the Bayesian analysis for treatment and control groups was discussed through the factorial designs by Kessler et al. (2020). Sokac et al. (2021) addressed the limitations of optimization and mathematical model for improving composting processes.

Heck, et al. (2021) outlined a thorough knowledge of Bayesian variable selection, Bayesian evaluation of cognitive models, and opportunities for Bayes Factor applications. Gardini et al. (2021) gave an idea on the log-transformation of a response variable by applying the Bayesian analysis of variance mixed models to examples and simulation datasets. Egburonu et al. (2021) discussed a balanced two-way analysis of variance of three cases such as the factors are fixed, random, and mixed by applying the Bayesian techniques. Gromping (2021) developed an algorithm for a two-level regular fractional factorial design with two-factor interactions. Ming-Chung Chang (2022) used the Bayesian approach for the minimum aberration criteria for many applications. Vijayaragunathan and Srinivasan (2022) discussed the comparison of Bayes Factors in both 2^3 and 2^4 full, fractional and reduced factorial designs.

The present study proposed considers a factorial design that includes all significant factors based on its strength by analysing full, fractional, and reduced factorial designs with three factors each at three levels adopting Bayesian approach Under Bayesian approach it is important to identify suitable priors to examine the strength/effectiveness of factors in these models. Thus, by identifying and quantifying the Bayes Factors through simulated datasets, the true weightage of the main/interaction effects in these three designs were discovered. Finally, the study concludes that the reduced factorial design produces better results than traditional one-third fractional factorial designs when there are no serious constraints in terms of time and resources for adding more factors to the model. The following section provides an algorithm for studying the strength of factors and then an illustration is provided for the same. The reliability of the results are studied through a simulation study and conclusions are provided at the end.

2. Algorithm for finding the strength of factors in 3^3 factorial design

In the 3^3 full, fractional, and reduced factorial designs, the following steps are to be followed to analyze, evaluate and identify the strength of factors.

Step 1: Consider the appropriate data for a 3^3 full factorial design and analyze the same to identify the significant main and interaction effects.

Step 2: Construct a fractional factorial design based on suitable confounding factors and then check its significance in the model.

Step 3: Generate a reduced factorial design according to the significant factors from the 3^3 full factorial design.

Step 4: Compute the Bayes Factor values for 3^3 full, fractional, and reduced factorial designs to compare the strength of factors while incorporating them into these designs.

Step 5: To extend the findings, the study employed simulated datasets to uncover the strength of factors through a variety of Bayesian priors and draw comprehensive and useful conclusions based on the strength of the factors.

The present research considers the basic 3^3 factorial experimental design to find the most significant factors in full factorial, fractional and reduced factorial designs. When the experimental run is large, usually a fractional factorial design is preferred which incorporates all major elements in the design. Let each factor be tested at three levels, with a_0, a_1, and a_2, is the three levels of a; b_0, b_1, b_2 being the three levels of b; and c_0, c_1, c_2 being the three levels of c, and so on. $[a_i]$ is the total number of treatment combinations with an a_i^{th} level. $[a_1]-[a_0]$ at the 0^{th} level, and $[a_2]-[a_1]$at the first level, there are two reactions to a unit amount of the factor. When one looks at a graph with the levels of factor a on the x-axis and the responses on the y-axis, one can see that when $[a_2]-[a_1]$ is almost equal to $[a_1]-[a_0]$, the responses will lie on a straight line, therefore, the linear effect of factor a is measured by the average of $[a_2]-[a_1]$ and $[a_1]-[a_0]$. When $[a_2]-[a_1]$ differs significantly from $[a_1]-[a_0]$, the replies will follow a parabola rather than a straight line. As a result, the study uses the difference between $[a_2]-[a_1]$ and $[a_1]-[a_0]$ to calculate the factor's quadratic effect. The linear and quadratic effects of a, indicated by $\mathrm{A_L}$ and $\mathrm{A_Q}$ respectively, are obtained by utilizing standard divisors to express the effects on a unit-based comparison as follows.

$$A_L = \frac{1}{r\,3^{n-1}}([a_2]-[a_0]) \qquad (1)$$

$$A_Q = \frac{1}{2r\,3^{n-1}}([a_2]-2[a_1]+[a_0]) \qquad (2)$$

Each treatment combination is replicated r times. The remaining elements for linear and quadratic effects can be defined in the same way. Any two-factor interaction has four degrees of freedom, which are classified as linear × linear, linear × quadratic, quadratic × linear, and quadratic × quadratic. The AB interaction can be divided into partitions as

$$A_LB_L = \frac{1}{2r\,3^{n-2}}(a_2-a_0)(b_2-b_0)$$

$$A_QB_L = \frac{1}{4r\,3^{n-2}}(a_2-2a_1+a_0)(b_2-b_0)$$

$$B_Q = \frac{1}{4r\,3^{n-2}}(a_2-a_0)(b_2-2b_1+b_0)$$

$$A_QB_Q = \frac{1}{8r\,3^{n-2}}(a_2-2a_1+a_0)(b_2-2b_1+b_0)$$

The total yields are substituted for the treatment combinations with the provided levels of a and b, and the multiplications are done as usual. Similarly, other two-factor interactions can be partitioned into single degrees of freedom. ABC is a three-factor interaction with 8 d.f. that can be partitioned into a single d.f. as illustrated below.

$$A_LB_LC_L = \frac{1}{4\,r\,3^{n-3}}(a_2-a_0)(b_2-b_0)(c_2-c_0)$$

$$A_LB_LC_Q = \frac{1}{8\,r\,3^{n-3}}(a_2-a_0)(b_2-b_0)(c_2-2c_1+c_0)$$

$$A_LB_QC_L = \frac{1}{8\,r\,3^{n-3}}(a_2-a_0)(b_2-2b_1+b_0)\,(c_2-c_0)$$

$$A_LB_QC_Q = \frac{1}{16\,r\,3^{n-3}}(a_2-a_0)(b_2-2b_1+b_0)(c_2-2c_1+c_0)$$

$$A_QB_LC_L = \frac{1}{8\,r\,3^{n-3}}(a_2-2a_1+a_0)(b_2-b_0)(c_2-c_0)$$

$$A_QB_LC_Q = \frac{1}{16\,r\,3^{n-3}}(a_2-2a_1+a_0)(b_2-b_0)(c_2-2c_1+c_0)$$

$$A_Q B_Q C_L = \frac{1}{16\, r\, 3^{n-3}} (a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)(c_2 - c_0)$$
$$A_Q B_Q C_Q = \frac{1}{32\, r\, 3^{n-3}} (a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)(c_2 - 2c_1 + c_0)$$

Where, the yields will now be substituted for the treatment combinations with the indicated levels of $a, b, c,$ and multiplications will be repeated as before. In a similar idea, higher-level interaction can be defined. The sum of squares of these effects and interactions was computed using standard principles.

Assume that in a factorial experiment, three components A, B, and C are being investigated, each with three levels. In the 3^3 factorial designs, there are 27 treatment combinations. Each main effect has two degrees of freedom, each two-factor interaction has four degrees of freedom, and the three-factor interaction has eight degrees of freedom. If there are $'n'$ replicates, there are $n\,3^3 - 1$ total degrees of freedom and $3^3(n-1)$ degrees of freedom for error. The sum of squares can be calculated using standard factorial design methods. Furthermore, if the factors are quantitative and each has one degree of freedom, the main effects can be partitioned into linear and quadratic components. Two-factor interactions can be decomposed into linear x linear, linear x quadratic, quadratic x linear, and quadratic x quadratic effects. Finally, the ABC three-factor interaction can be divided into eight single degrees of freedom components, such as linear x linear, linear x quadratic, quadratic x quadratic, and so on. Three-factor interaction isn't very useful in general. However one can separate the I and J components from the three-factor interaction in order to get two-factor interactions. AB, AB^2, AC, AC^2, BC, and BC^2 are the two-factor interactions, and each component has two degrees of freedom. These components have no physical meaning. The W, X, Y, and Z components of the three-factor interaction ABC are the four orthogonal two-degrees of freedom components of the interaction. The ABC interaction components are referred to as AB^2C^2, AB^2C, ABC^2, and ABC, respectively. The I, J, W, X, Y, and Z components have no practical interpretations. The design is built using the notations listed below.

$$W(ABC) = AB^2C^2;\ X(ABC) = AB^2C;\ Y(ABC) = ABC^2 \text{ and } Z(ABC) = ABC$$

In the construction of fractional factorial design for the 3^3design, one can use any of the ABC interaction components, such as $ABC, AB^2C, ABC^2,$ and AB^2C^2. The 3^3 factorial confounded in three blocks of nine runs each. Thus, the possible components of interaction contrast as given below have 12 different one-third fractions.

$L = x_1 + x_2 + x_3 = u\ (mod\ 3)$ for ABC,
$L = x_1 + 2x_2 + x_3 = u\ (mod\ 3)$ for AB^2C
$L = x_1 + x_2 + 2x_3 = u\ (mod\ 3)$ for ABC^2
and $L = x_1 + 2x_2 + 2x_3 = u\ (mod\ 3)$ for AB^2C^2,
where $u = 0, 1\ or\ 2$,

If the number of significant factors in the full factorial design is more than the number of significant factors in the fractional factorial design, a reduced factorial design may be the better option. The purpose of this study is to create a reduced factorial design using only significant factors. It cannot be anticipated until the full factorial design is performed. If the experimenter wants that no information should be lost throughout the design and also there is no constraint to include all of the main and interaction factors except the non-significant factors then a reduced factorial design is more valuable and informative.

This study employed five alternative priors to obtain the Bayes Factors for full, fractional, and reduced factorial designs. Maruyama (2009), Wetzels et al. (2012), Wang and Sun (2014), and Wang et al. (2015) have all examined Bayes Factors conceptually. These priors are discussed by Vijayaragunathan and Srinivasan (2020, 2022) in their study of hierarchical two-way ANOVA models and factorial designs of factors each at two levels.

a). Zellner's g Prior

Prior to Bayesian hypothesis testing, Zellner's priors were most widely utilized. Many authors have previously examined this, including George and Foster (2016), Kass and Wasserman (1995), and others. By changing the value of g, one can evaluate two priors: (a) Unit Information Prior (UIP) if $g = n$, and (b) Risk Inflation Criterion (RIC) if $g = k^2$, where n=number of observations and k= number of predictors in the regression model. The Bayes Factor for the full model to the null model is

$$BF = (1+\mathrm{g})^{(n-k-1)/2}[1+g(1-R^2)]^{-(n-1)/2} \tag{3}$$

b). Jeffreys-Zellner-Siow Prior

In the Jeffreys-Zellner-Siow (JZS), one estimates g from the data. This prior is a mixture of priors discussed by Liang et al. (2008). The following equation gives the Bayes Factor for the full model to the null model

$$BF = \frac{\left(\frac{n}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty (1+g)^{(n-k-1)/2}\,[1+g\,(1-R^2)]^{-(n-1)/2}\, g^{-3/2}\, e^{-n/2g} d \tag{4}$$

c). Hyper-g Prior

A family of prior distributions on g is known as the hyper- g prior. The term a range from 2 to 4, resulting in distinct hyper- g prior behaviour. For simplicity, one uses only two values: $a = 3$ and $a = 4$. The equation given below is the Bayes Factor for the full model to the null model.

$$BF = \frac{a-2}{2}\int_0^\infty (1+g)^{\frac{n-k-1-a}{2}}\,[1+g(1-R^2)]^{-\frac{n-1}{2}} dg \tag{5}$$

3. Illustration

The present research considers the illustration from Montgomery (2019),

"A machine is used to fill 5-gallon metal containers with soft drink syrup. The variable of interest is the amount of syrup loss due to frothing. Three factors are thought to influence frothing: the nozzle design (A), the filling speed (B), and the operating pressure (C). Three nozzles, three filling speeds, and three pressures are chosen and two replicates of a 3^3 factorial experimental run".

The ANOVA output for syrup loss data is presented in Table 1 and it is observed that the filling speed B and the operating pressure C are statistically significant. The two-factor interactions, AB, AC^2, BC, BC^2, and three-factor interactions AB^2C^2 are also significant. It shows that the interaction effects are influenced by the soft drink syrup loss data even if the main effect, nozzle design A, is not significant in the full factorial design.

Table 1. ANOVA output for 3^3 full factorial design

Source of Variation	Sum Sq.	Df	Mean Sq.	F value	Pr(>F)	
factor(A)	994	2	497	1.165	0.327102	
factor(B)	61190	2	30595	71.735	1.57e-11	***
factor(C)	69105	2	34553	81.014	3.89e-12	***
factor(AB)	6174	2	3087	7.238	0.003042	**
factor(AB^2)	127	2	63	0.149	0.862592	
factor(AC)	635	2	318	0.745	0.484444	
factor(AC^2)	6879	2	3439	8.064	0.001795	**
factor(BC)	8581	2	4291	10.060	0.000543	***
factor(BC^2)	4273	2	2136	5.009	0.014116	*
factor(ABC)	19	2	9	0.022	0.978244	
factor(AB^2C)	222	2	111	0.260	0.772962	
factor(ABC^2)	584	2	292	0.685	0.512748	

factor(AB^2C^2)	3804	2	1902	4.460	0.021207	*
Residuals	11515	27	426			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

The study proceeds to employ a 3^3 factorial design with three blocks of nine runs each. When the three-factor interaction AB^2C^2 will be confounded by blocks, let $L = x_1 + 2\,x_2 + 2x_3$ the defining contrast, then the treatment combinations belonging to the principal block are $000, 012, 101, 202, 021, 110, 122, 211, 220$.

Also other treatment combinations $200, 212, 001, 102, 221, 010, 022, 111, 120$ and $100, 112, 201, 002, 121, 210, 222, 011, 020$ are belonging to block 2 and 3 respectively.

Table 2. ANOVA output for 3^{3-1} fractional factorial design (AB^2C^2 at 0)

Source of Variation	Sum Sq.	Df	Mean Sq.	F value	Pr(>F)	
factor(A)	2722	2	1361	3.091	0.095065	
factor(B)	9855	2	4928	11.192	0.003621	**
factor(C)	23211	2	11605	26.359	0.000173	***
factor(AB)	5567	2	2783	6.322	0.019279	*
Residuals	3962	9	440			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Table 3. ANOVA output for 3^{3-1} fractional factorial design (AB^2C^2 at 1)

Source of Variation	Sum Sq.	Df	Mean Sq.	F value	Pr(>F)	
factor(A)	1519	2	760	1.120	0.367777	
factor(B)	22663	2	11332	16.708	0.000934	***
factor(C)	23644	2	11822	17.431	0.008030	***
factor(AB)	594	2	297	0.438	0.658385	***
Residuals	6104	9	678			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Table 4. ANOVA output for 3^{3-1} fractional factorial design AB^2C^2 at 2

Source of Variation	Sum Sq.	Df	Mean Sq.	F value	Pr(>F)	
factor(A)	5352	2	2676	16.62	0.00095100	***
factor(B)	36134	2	18067	112.22	0.00000043	***
factor(C)	22599	2	11299	70.18	0.00000324	***
factor(AB)	4924	2	2461	15.28	0.00012770	**
Residuals	1449	9	161			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

The ANOVA outputs for these one-third fractional factorial designs are shown in Tables 2 - 4. The main effects B and C, and the interaction effect AB are significant in the fractional factorial design when the factor AB^2C^2 at levels 0 and 1 (see Tables 2 and 3). But, three main effects and interaction AB are significant in the fractional factorial design when the factor AB^2C^2 at level 2 (see Table 4). The main effect A performs well when it interacts with another factor, a synergetic effect, in this experimentation. Moreover, all the fractional factorial designs do not provide the same results even if the factors are the same in the ANOVA output.

Unlike the main effect, nozzle design A significantly differs on the soft drink syrup only when the interaction AB^2C^2 at level 2. The main effects, filling speed B and operating pressure C, significantly differ in all levels of AB^2C^2. Also, the nozzle design interaction with different levels of filling speeds determined the loss of syrup impact in all three fractional factorial designs.

Table 5. ANOVA output for 3^3 reduced factorial design

Source of Variation	Sum Sq.	Df	Mean Sq.	F value	Pr(>F)	
factor(A)	6008	2	3004	9.942	0.000665	***
factor(B)	42010	2	21005	69.520	6.13e-11	***
factor(C)	50480	2	25240	83.537	8.53e-12	***
factor(AB)	6698	4	1675	5.542	0.002459	**
factor (BC)	10532	4	2633	8.715	0.000151	***
factor(AC^2)	1674	1	1674	5.540	0.026748	*
factor(AB^2C^2)	2798	1	2798	9.262	0.005440	**
Residuals	7554	25	302			

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

In the fractional factorial design, this study considered one-third of the observations from the full factorial design. Since one may lose potentially significant interaction effects in the fractional factorial designs, instead of ignoring factors the present study incorporated all the significant factors in the proposed reduced factorial design. . The ANOVA output for the reduced factorial design is presented in Table 5 and it is observed that the main effect A is highly significant in the reduced factorial design.

Table 6. Bayes Factor values for 3^3 full, fractional, and reduced factorial designs

Prior	Full Facto-rial Design	One-third Fractional Factorial Design for the three-factor interaction												Re-duced Factorial Design
		ABC			AB^2C			ABC^2			AB^2C^2			
		0	1	2	0	1	2	0	1	2	0	1	2	
Zellner’s g prior (UIP)	5.60	-0.34	-4.34	-1.12	-1.44	-0.18	-4.36	-0.56	0.001	-2.18	-1.57	-13.02	-10.11	4.58
Zellner’s g prior (RIC)	4.67	-5.12	-10.88	-6.54	-7.05	-4.80	-17.83	-5.55	-4.42	-8.16	-7.26	-38.18	-32.08	4.04
Jeffreys-Zellner-Siow	6.80	-0.27	-2.41	-0.71	-0.88	-0.17	-2.56	-0.39	-0.06	-1.26	-0.94	-5.89	-5.36	5.17
Hyper-g prior (a=3)	7.03	0.15	-0.58	-0.03	-0.09	0.20	-0.41	0.10	0.25	-0.23	-0.12	-1.23	-1.19	5.27
Hyper-g prior (a=4)	6.83	0.22	-0.35	-0.08	-0.04	0.26	-0.07	0.18	0.29	-0.40	0.02	-0.95	-0.91	5.07

The main effect A is one of the highly significant factors in the reduced factorial design because the combination of other factor effects will influence the main effect A in the reduced factorial design. The ANOVA table for full, fractional and reduced factorial design provides slight variation in the results. Therefore, as proposed the study proceeded with the Bayesian Approach for quantifying the results in a better way. The Bayes Factor values for the full factorial, 12 different one-third fractional factorial, and reduced factorial designs are shown in Table 6. The factors support 5 to 7 times the model in a full factorial design. One may use any of the ABC interaction effects for further comparison.

The study used AB^2C^2 for computing the Bayes Factors for all factors from the one-third fractional factorial design. The results revealed that data support the null model for some cases “Poorly” and others “Strongly”, which indicates that data does not support the fractional factorial design. In the fractional factorial design, priors like Zellner's g prior (UIP), Jeffrey-Zellner-Siow prior, and both Hyper-g priors offer more or less comparable results. However, the Bayes Factor values for Zellner's g prior (RIC) are nearly 3 to 6 times smaller than those for the other prior. Moreover, the Bayes Factors for reduced factorial design results are the same as the full factorial design. In order to generalize the findings, the study proceeded to determine the Bayes Factor values for three different factorial design variations. In the following section, the study compared the Bayes Factor values for these designs for simulated data.

4. Simulation Study

In order to get reliable results, dataset was simulated for the respective designs. Further, to calculate the values of the Bayes Factors for five priors, the simulation data was run over 10,000 iterations with an error variance of 1. The Bayes Factors were calculated for various datasets with error variances of 5, 25, and 50, respectively. The five prior Bayes Factors for these simulated data to the 3^3 full factorial, one-third fractional factorial, and reduced factorial designs are shown in Figures 1-5. The mean and standard deviation of Bayes Factor values for 3^3full, fractional and reduced factorial designs were presented in Table 7.

Table 7. Average (Standard Deviation) of 10000 Bayes Factor values to the simulated datasets of 3^3 full, 3^{3-1} fractional, and reduced factorial designs for five priors when the error variances are 1, 5, 25 and 50.

Error Variance (σ_e^2)	Zellner's g-prior (UIP)	Zellner's g-prior (RIC)	Jeffreys -Zellner -Siow	Hyper-g prior (a=3)	Hyper-g prior (a=4)
			3^3 full factorial design		
1	5.12 (2.13)	1.17 (2.28)	6.43 (1.69)	6.74 (1.46)	6.54 (1.41)
5	5.07 (2.16)	1.12 (2.32)	6.39 (1.72)	6.70 (1.48)	6.51 (1.43)
25	3.77 (3.03)	0.56 (3.24)	5.36 (2.41)	5.86 (2.03)	5.71 (1.95)
50	1.78 (3.77)	0.15 (3.99)	3.79 (2.99)	4.63 (2.40)	4.53 (2.03)
			3^{3-1} fractional factorial design (AB^2C^2 at 0)		
1	-1.11 (1.51)	-6.12 (2.72)	-0.64 (0.89)	0.04 (0.36)	0.13 (0.27)
5	-1.22 (1.51)	-6.35 (2.59)	-0.71 (0.86)	0.01 (0.34)	0.11 (0.26)
25	-2.95 (1.39)	-9.11 (1.92)	-1.67 (0.74)	-0.34 (0.23)	-0.17 (0.19)
50	-4.68 (1.42)	-11.28 (1.74)	-2.60 (0.78)	-0.61 (0.21)	-0.39 (0.18)
			3^{3-1} fractional factorial design (AB^2C^2 at 1)		
1	-12.47 (1.43)	-36.99 (2.55)	-5.81 (0.32)	-1.23 (0.03)	-0.95 (0.03)
5	-12.57 (1.47)	-37.19 (2.47)	-5.83 (0.32)	-1.23 (0.03)	-0.95 (0.03)
25	-13.02 (0.85)	-39.21 (1.93)	-6.13 (0.36)	-1.26 (0.03)	0.09 (0.04)
50	-15.43 (1.36)	-41.28 (1.71)	-6.60 (0.44)	-1.30 (0.04)	-1.02 (0.04)
			3^{3-1} fractional factorial design (AB^2C^2 at 2)		
1	-10.04 (0.62)	-31.31 (2.35)	-5.35 (0.09)	-1.19 (0.01)	-0.91 (0.01)
5	-10.20 (0.65)	-31.92 (2.24)	-5.38 (0.10)	-1.19 (0.01)	-0.92 (0.01)
25	-12.31 (1.05)	-36.89 (1.84)	-5.76 (0.22)	-1.22 (0.02)	-0.95 (0.02)
50	-14.59 (1.32)	-40.24 (1.75)	-6.34 (0.39)	-1.28 (0.04)	-0.99 (0.04)
			3^3 reduced factorial design		
1	4.18 (1.65)	2.01 (1.82)	4.86 (1.31)	5.03 (1.11)	4.85 (1.06)
5	4.10 (1.73)	1.94 (1.73)	4.80 (1.37)	4.98 (1.16)	4.80 (1.10)
25	2.53 (2.54)	0.77 (2.77)	3.58 (1.99)	4.00 (1.62)	3.88 (1.54)
50	0.23 (3.15)	-1.22 (3.8)	1.79 (2.45)	2.69 (1.83)	2.65 (1.71)

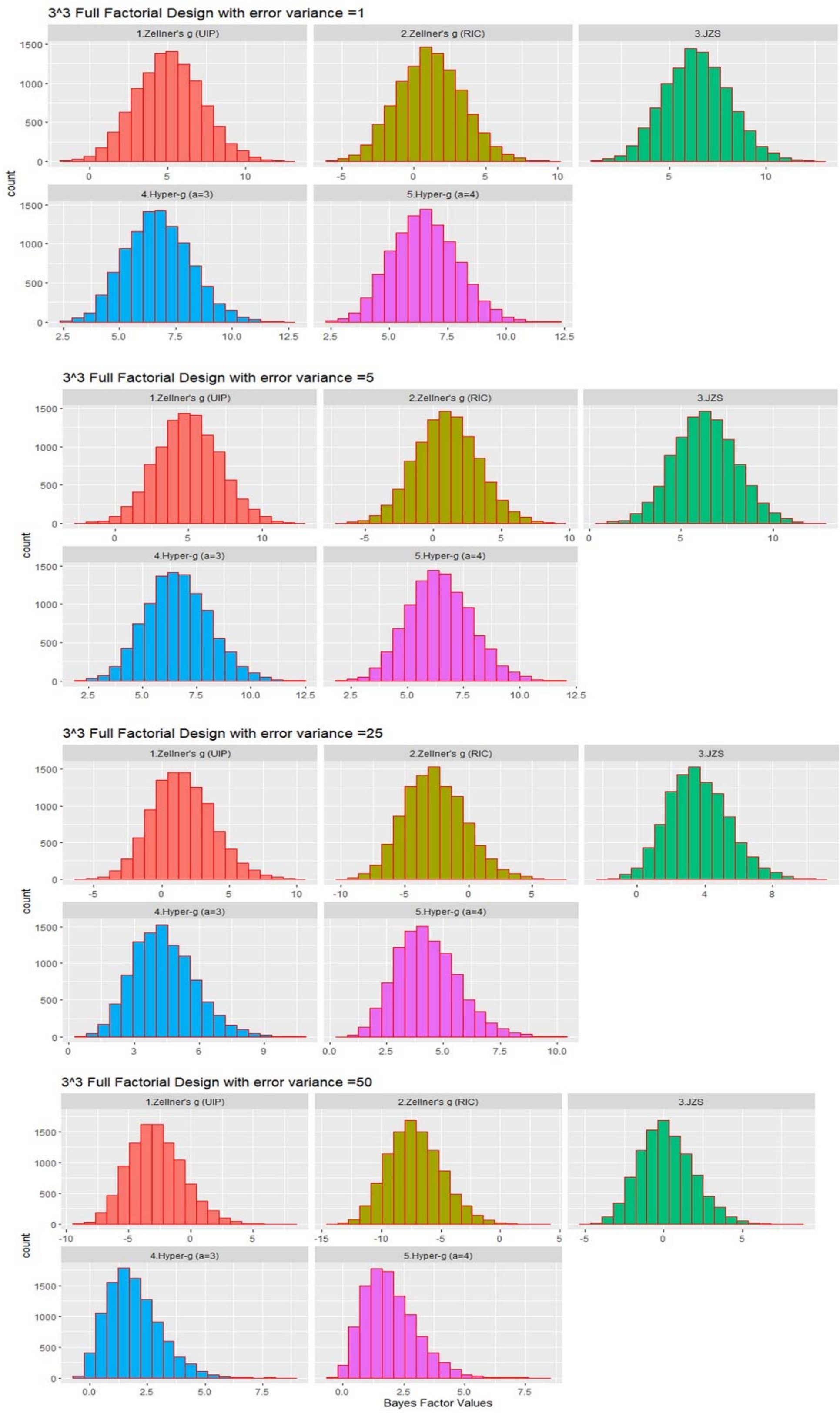

Figure 1. Bayes Factor values for 3^3full factorial design to the different simulation datasets

The results of 3^3 full factorial design are obtained from Figure 1, it may be seen that in Zellner's g (UIP) the distribution of the Bayes Factor ranges between 0 and 10, which means that the simulated dataset with an error variance of 1 for the 3^3 full factorial design data supports the model specified 0 to 10 times. The data support the model 0 to 5 times when an error variance is 5. The 3^3 full factorial design with error variance 25 provides mixed results that, out of 10000 iterations, approximately half of the iterations, the data supported the null model and the remaining half supported the specified model. Also, when the error variance is increased to 50, it provides mixed results.

For the Zellner's g prior (RIC) the data supports the model -5 to 7 times and -5 to 7 times when error variances are 1 and 5. Thus, half of the iteration result supports the null model and the remaining iterations support the full model. When error variance is 25, the data mostly supports the null model and sometimes supports the full model also. When error variance is 50, the data support the null model -12 to 0 times.

For the JZS the data supports the full factorial model 0 to 11 times when error variances are 1 and 5; the Bayes Factor values range between -1 and 9 when error variance is 25, almost the data supports the full model; when error variance is 50, the Bayes Factor values between -5 and 5 which mean that half of the result of the iteration supports the null model and remaining iterations supports the full model.

Both Hyper-g priors provide similar results such that the data supports the model 2 to 11 times, 2 to 11 times, 1 to 9 times and 0 to 5 times when error variances are 1, 5, 25, and 50 respectively. Thus, the two Hyper-g priors always support the 3^3 full factorial model.

The average of the Bayes Factor values for Zellner's g (UIP) prior are 5.12, 5.07, 3.77, and 1.78; for Zellner's g (RIC) prior are 1.17, 1.12, 0.56, and 0.15; for JZS prior are 6.43, 6.39, 5.36, and 3.79; Hyper-g (a=3) prior are 6.74, 6.70, 5.86, and 4.63; for Hyper-g (a=4) prior are 6.54, 6.51, 5.71, and 4.53 for the simulated dataset of 3^3 full factorial design with error variances of 1, 5, 25 and 50 respectively which are presented in Table 7.

The results of 3^{3-1} fractional factorial design (AB^2C^2 at 0) are obtained from Figure 2 and it may be seen that in Zellner's g (UIP) the distribution of the Bayes Factor for 3^{3-1} fractional factorial design (AB^2C^2 at 0) ranges between -5 and 2, -5 and 2, -7.5 and 0, and -9 and 0; for Zellner's g (RIC) prior ranges between -10 and 1, -10 and 1, -15 and -3, and -16 and -4; for JZS prior ranges between -3 and 2, -3 and 2, -4 and 0, and -5 and 0; both Hyper-g priors have almost same ranges between -1 and 1, -1 and 1, -1 and 0.5, and -1.2 and 0.3 when the error variance are 1, 5, 25, and 50 respectively.

The results of 3^{3-1} fractional factorial design (AB^2C^2 at 1) are obtained from Figure 3 and it may be seen that in Zellner's g (UIP) the distribution of the Bayes Factor for 3^{3-1} fractional factorial design (AB^2C^2 at 1) ranges between -16 and -9, -17 and -10, -17 and -11, and -18 and -11; for Zellner's g (RIC) prior ranges between -42 and -30, -45 and -30, -45 and -32, and -45 and -35; for JZS prior ranges between -7 and -4, -7 and -5, -7 and -5, and -8 and -5; both Hyper-g priors have almost same ranges between -1.3 and -1, -1.4 and -1.1, -1.4 and -1.2, and -1.4 and -1.2 when the error variance are 1, 5, 25, and 50 respectively.

The results of 3^{3-1} fractional factorial design (AB^2C^2 at 2) are obtained from Figure 4 and it may be seen that in Zellner's g (UIP) the distribution of the Bayes Factor for 3^{3-1} fractional factorial design (AB^2C^2 at 2) ranges between -12 and -9, -13 and -9, -15 and -9, and -18 and -10; for Zellner's g (RIC) prior ranges between -37 and -25, -37 and -25, -42 and -30, and -45 and -35; for JZS prior ranges between -5.75 and -5.25, -6 and -5, -7 and -5, and -8 and -5; for Hyper-g (a=3) prior ranges between -1.22 and -1.18, -1.23 and -1.17, -1.3 and -1.2, and -1.4 and -1.2; for Hyper-g (a=4) prior ranges between -1 and -0.9, -0.96 and -0.12, -1.05 and -0.9, and -1.1 and -0.9 when the error variance are 1, 5, 25, and 50 respectively.

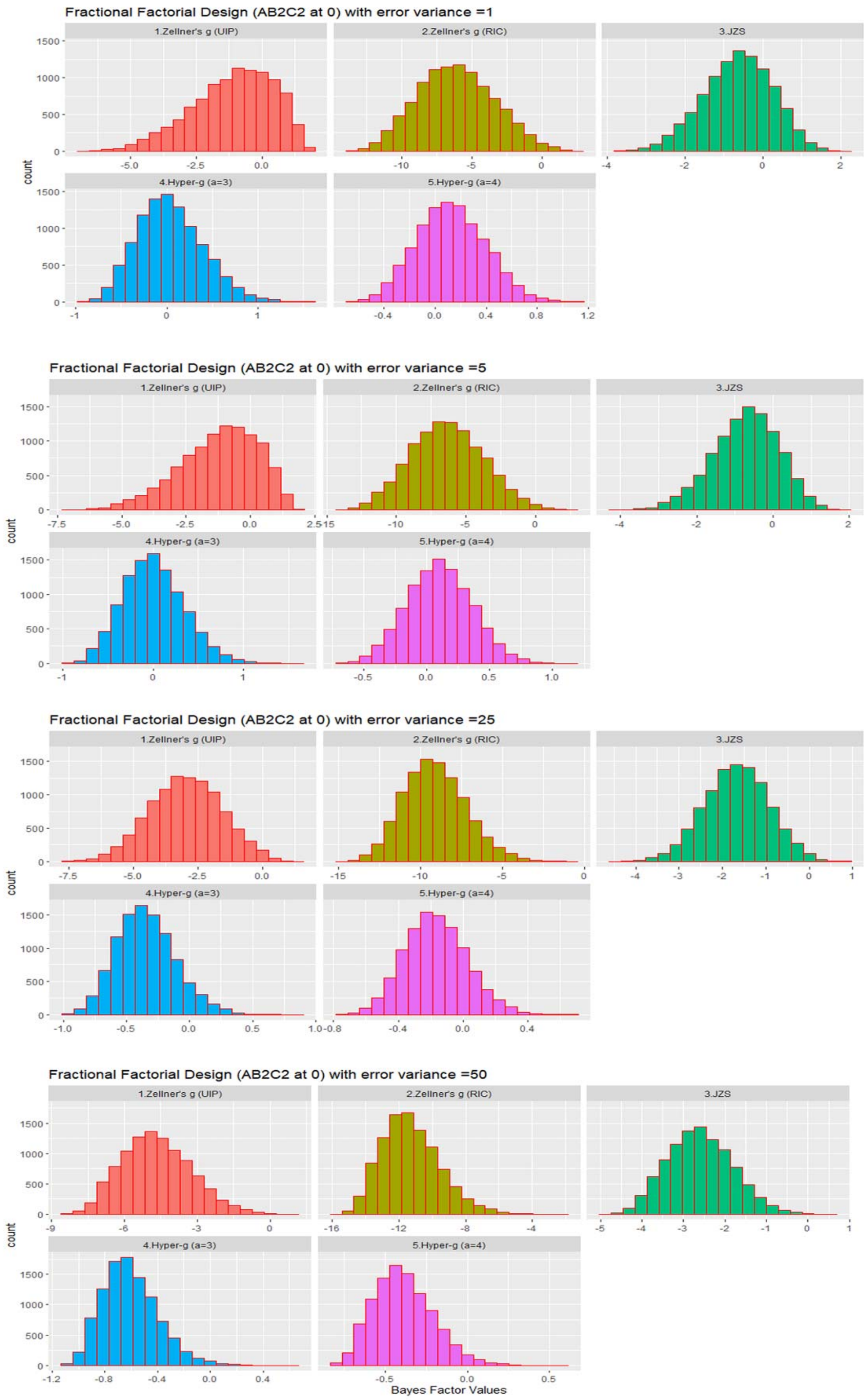

Figure 2. Bayes Factor values for 3^{3-1}fractional factorial design $(AB^2C^2\ at\ 0)$ to the different simulation datasets

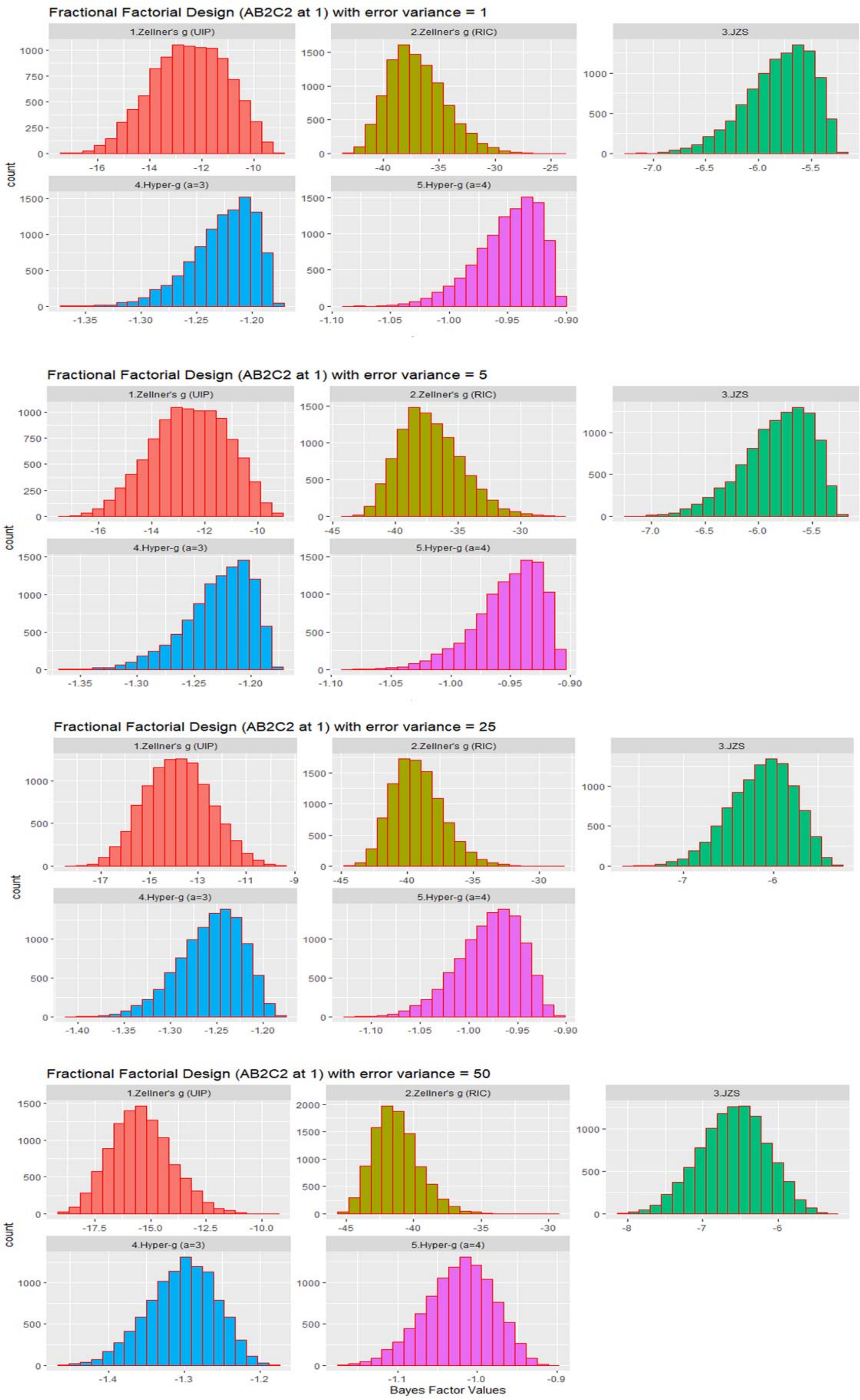

Figure 3. Bayes Factor values for 3^{3-1}fractional factorial design $(AB^2C^2\ at\ 1)$ to the different simulation datasets

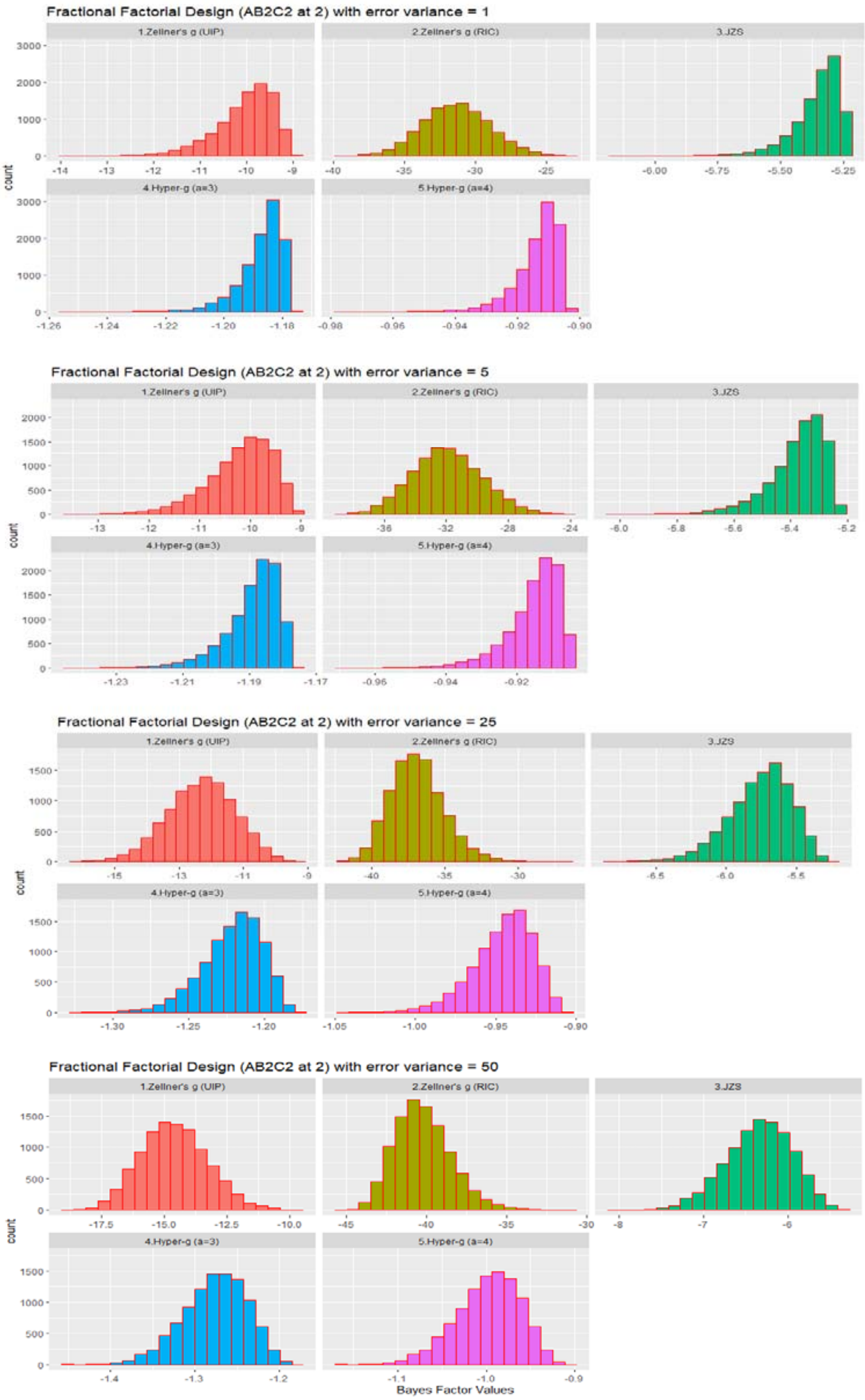

Figure 4. Bayes Factor values for 3^{3-1}fractional factorial design $(AB^2C^2\ at\ 2)$ to the different simulation datasets.

The average of the Bayes Factor values for Zellner's g (UIP), Zellner's g (RIC), Jeffreys-Zellner-Siow, Hyper-g (a=3) and Hyper-g (a=4) priors are presented in Table 7.

Thus, the distribution of the Bayes Factor values show that the simulated dataset almost "Decisively" supports all 3^{3-1} fractional factorial designs invariably in the present study. Particularly, the Bayes Factor values for both the Hyper-g priors have less variability than the other prior and these priors support "Poorly" the null model.

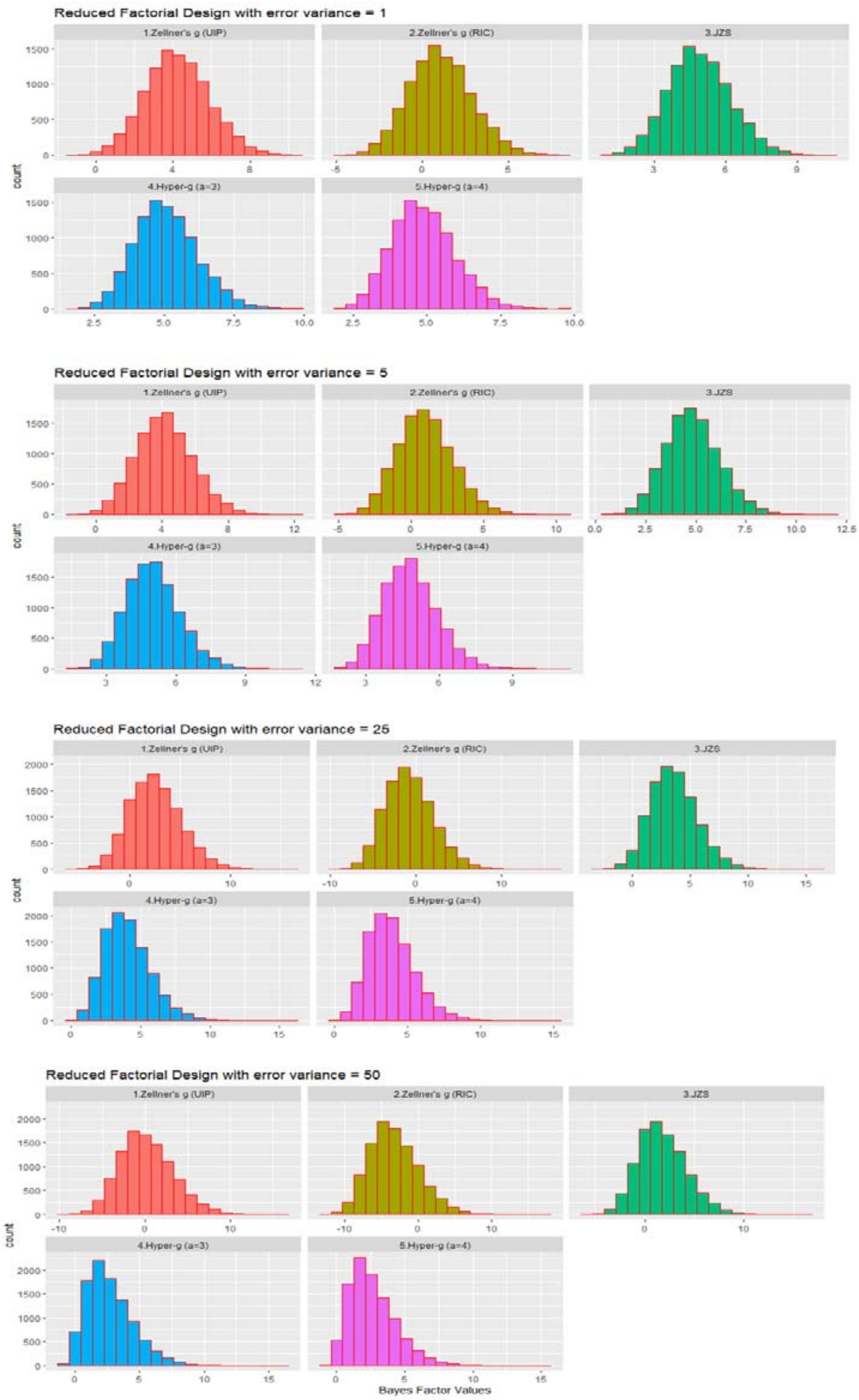

Figure 5. Bayes Factor values for 3^3 reduced factorial design to the different simulation datasets

From Figure 5, it may be seen that in Zellner's g (UIP) the distribution of the Bayes Factor ranges between 0 and 9, which means that the simulated dataset with an error variance of 1 for the 3^3 reduced factorial design data supports the model specified 0 to 9 times. The data support the model 0 to 12 times when an error variance is 5; the data support the model -1 to 10 times when an error variance is 25; The 3^3 reduced factorial design with an error variance of 50 provides mixed results that, the data support the model -10 to 10 times. Out of 10000 iterations, approximately half of the iterations, the data supported the null model and the remaining half supported the specified model. For the Zellner's g prior (RIC) the distribution of the Bayes Factor for 3^3 reduced factorial design ranges between -5 and 5, -5 and 10, -10 and 10, and -10 and 5; for JZS prior ranges between 4 and 9, 2 and 9, 0 and 10, and -2 and 10; both Hyper-g priors have almost same ranges between 2 and 8, 3 and 8, 0 and 10, and 0 and 10 when the error variance of 1, 5, 25, and 50 respectively. Thus, all priors except Zellner's g (RIC) are provided "Decisively" to support the reduced factorial model. Particularly, Zellner's g (RIC) prior gives mixed results that more or less half of the iterations supported the null model and the remaining half supported the reduced factorial model invariably among the different simulated datasets with various error variances.

The average of the Bayes Factor values for Zellner's g (UIP) prior are 4.18, 4.10, 2.53, and 0.23; for Zellner's g (RIC) prior are 2.0, 1.94, 0.77, and -1.22; for JZS prior are 4.86, 4.80, 3.58, and 1.79; Hyper-g (a=3) prior are 5.03, 4.98, 4.00, and 2.69; for Hyper-g (a=4) prior are 4.85, 4.80, 3.88, and 2.65 for the simulated dataset of 3^3 reduced factorial design with error variances of 1, 5, 25 and 50 respectively which are presented in Table 7.

The Bayes Factor values of different priors for the simulated datasets obtained the following results. In general, among the five priors, Zellner's g prior (RIC) produces a much smaller average of Bayes Factor values against the simulated datasets as compared with all other priors. This is because this prior has a high value of g, which is the square of the number of predictors in the respective model. The same results are obtained by RIC prior in all fractional factorial designs as well as the reduced factorial design. Particularly, all the one-third fractional factorial designs support the null model invariably. Furthermore, both the Hyper-g priors have a less standard deviation of the Bayes Factor values compared to all other priors. Finally, the Bayes Factor values for the reduced factorial design are almost close to the Bayes Factor values of full factorial design.

5. Summary and Conclusion

In this study, the investigation is basically on the use of Bayesian measures to determine the strength of the factors in the 3^3 factorial design. The Bayesian framework has been widely used in model selection, here the Bayesian principle was used to determine the intensity of the factors in 3^3 full, fractional, and reduced factorial models. Based on the classical factorial design analysis, it is found that the main effects B, and C; the first-order interaction AB, BC, AC^2 and BC^2; the second-order interaction AB^2C^2 are significant in the 3^3 full factorial design. The main effects B and C, and the interaction AB are significant in all three possible one-third fractional factorial designs. But all the factors are significant in fractional factorial design when the factor AB^2C^2 at level 2. All the factors are significant in 3^3 reduced factorial design. Furthermore, all the priors do not contribute the same Bayes Factor values to the respective factorial designs. Based on the Bayes Factor values, the factors supported 5 to 7 times the 3^3 full factorial design. In the simulation study, the Bayes Factor values for full and reduced factorial designs are positive which means that the data "Decisively" support the respective models. But, the Bayes Factors for fractional factorial designs are negative, which shows that the data does not support the fractional factorial designs. All these three one-third fractional factorial models do not produce similar results and also do not resemble the full factorial design in the results. In the proposed model, the Bayes Factor values in the reduced factorial design show "Strong" support for the model, the same as the full factorial design. Finally, it is concluded and generalized that the results based on the real-life application and simulated dataset, that the reduced factorial design is a more appropriate model for the full factorial design compared to the fractional factorial designs. Hence, the reduced factorial design is a better alternative to the full factorial design to check the strength or intensity of the factors. Furthermore, it is proposed to apply the same technique for more than three factors each at three levels of factorial designs, Analysis of the Covariance model, split-plot design, asymmetrical designs etc. to find the strength/intensity of the factors in the respective models in the future studies.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Author's Contributions

Authors contributed equally and approved the final manuscript.

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