MEASURING RISK PROFILE WITH A MULTIDIMENSIONAL RASCH ANALYSIS[1]

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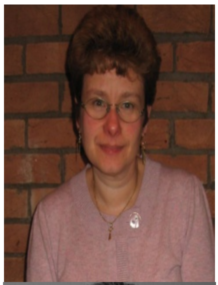

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Abstract: *In this paper we propose an evaluation of investors' risk profiles such as to meet the minimal requirements that Italian financial institutions must satisfy by law (d. lgs. 164, 2007). Thus we investigate all aspects specific to so-called risk profiles: an investor's knowledge and his financial experience (concerning financial instruments and their use); financial objectives, a personal predisposition to risk /earn and the temporal horizon.*

The methodology used in financial literature with regard to risk profiles is essentially based on simplistic statistical analyses that often fail to consider possible psychological aspects. In order to account for investor preferences and psychological attitudes, we suggest to use an item response theory model. We first assume a unidimensional model, belonging to the family of Rasch models and then, as an alternative approach, a Generalized Multidimensional Rasch model. In particular, the objective is to assess the value of a questionnaire whose items describe different characteristics of the main latent variable risk profile.

Under the assumption of a multidimensional measurement model, given the multivariate position of each investor with respect to identified latent traits we can represent his position with respect to possible investments proposed by a bank and we can identify different situations that respect the investor's risk profile and best characterize typical investor choices.

Key words: *risk profile; predisposition to risk/earn; Rasch model; multidimensional IRT models; marginal maximum likelihood*

1. Introduction

With the advent of the new technology and the possibility to invest on-line, the number of personal investments has grown substantially in the last decade. For these reasons, many operators have launched and organized web sites where simple and intuitive operations supply investors with suggestions as to potential investment strategies. In the context of investment advice to individual investors, several financial institutions in Italy, Canada, the Netherlands, New Zealand, and the United States have started to use risk profiles of their clients. These risk profiles used by different banks all over the world are standard questionnaires for potential clients. By applying appropriate risk profiling the institution can help potential clients make decisions that meet the personal predisposition to risk/earn. In some countries (for example, in Italy and the Netherlands) financial institutions are obliged by law to construct risk profiles that include questions on the investors' knowledge of the financial instruments, on the investors' time horizon and on their risk preferences. We generally refer to these three aspects as the main characteristics of a unique latent trait that we call the predisposition to risk/earn. This information is generally analyzed to determine the most suitable strategy and the best investment options available for clients. The idea behind the questions on investors' knowledge of the financial instruments is consistent with the responsibility of the financial institution to propose more transparent and less risky investments for clients with a lower level of knowledge. Questions on the investors' time horizon serve to address choices in stocks for investors with longer time horizons and on bonds for investors with a shorter time horizon (see, among others, Barberis, 2000). Similarly, questions on risk preferences should direct investors most tolerant to risk towards more aggressive and riskier financial instruments (contingent claims and stocks) and the most risk adverse investors towards the most conservative and less risky assets (bonds and treasury bills). Therefore, risk profile questionnaires are of help to financial institutions in proposing an initial pre-selection of capital in different financial instruments to potential clients (see Veld and Veld-Merkoulova 2008). Clearly this initial portfolio preselection is consistent with investor preferences as summarized in the questionnaire. For this reason, the same information derived from the combined score from knowledge, time horizon and risk preference questions should also be used in determining proper portfolio strategy. On the other hand, in modern portfolio theory, optimal choices are essentially based on their consistency with respect to a given stochastic ordering (see Ortobelli *et al.*, 2009). Thus the optimality of the choices in such portfolio theory is essentially based on an implicit assumption as to the investor's behavior.

Unlike classic portfolio theory, in this paper we propose a methodology for identifying an investor's particular attitude to taking risks/earning on the market, based on the analysis of a risk profile questionnaire. Moreover, in this paper we show how the results can be used to propose a preselection of optimal choices. Since the main objective of risk profiling consists in determining investors' preferences and their personal predisposition to risk/earn, we have to account for various psychological aspects. We observe that in the recent literature the analysis of investor risk perceptions is essentially based on the statistical valuation of the results in each single question (see Veld and Veld-Merkoulova, 2008). Doing so we partially lose the opportunity to value and measure the psychological aspects intrinsic in the questionnaire, that can be highlighted using psychometric models.

In the psychometric field, the Rasch analysis is the one of the more frequently used tool for evaluating latent variables. A very important feature of the Rasch analysis is that it allows an interactive control of the fit of the model to the data. Moreover, when the data fit the model, we obtain estimates that represent interval measurements. For these reasons, in this paper we submit the results of a questionnaire to a Rasch analysis. Georg Rasch (Rasch, 1960; Fischer and Molenaar, 1995) developed a mathematical model for constructing measures based on the responses of a number of subjects to the same set of items. The responses to the items are typically scored with zero and positive integers, i.e. 0, 1, 2,..., k (ordered categories), to represent the increasing level of the response on some latent trait variable. The so called *Rasch model*, refers to the case of 2 ordered categories (k=1). In this paper we shall refer to the case of k >1 (polytomous case). According to a polytomous Rasch model the probability each score is a function of the difference between two parameters: 1) a person parameter, often called *ability* (or *trait*, or *proficiency*; examples of hypothetical traits are skills, attitudes, preferences and achievements), and 2) an item parameter, which in turn is the sum of an item location parameter (often called item *difficulty*) and a threshold parameter. By *ability* we refer to the level of the predisposition to risk/earn. So a high value for a person parameter means a high level of tolerance to risk (i.e. a risk taking subject), while low values of the latent trait identify a subject adverse to risk. Under model expectations, for every item a person with higher ability always has a higher probability of endorsement, or success, than a person with lower ability. Likewise, a more difficult item always has a lower probability to be endorsed by respondents, regardless of individual ability.

The more simple assumption underlying most tests is that the latent trait is a scalar (unidimensional model). Besides, sometimes we may also assume that the response to an item is governed by a multidimensional latent trait. Now, our original questionnaire presents three "possible" dimensions (knowledge, risk preferences, time horizon), by construction, then both the approaches are feasible. In this paper both approaches to the analysis are considered comparatively.

Comparison with the unidimensional analysis shows the advantages of the multidimensional approach for interpreting the dataset. The empirical results resulting from this Rasch analysis serve to formulate an initial preselection with respect to possible investments proposed by the bank.

In the next section, we describe the questionnaire and the sample used in the analysis. In section 3, we consider a unidimensional approach to the answers to the questionnaire. In section 4, we extend the analysis to a multidimensional approach. In the last section, we provide concluding remarks regarding the potential applications of the paper's findings.

2. Data source

To assess the risk profile in this preliminary analysis, we decided to give a simple questionnaire to a sample of Economics and Business Administration students from the University of Bergamo. Therefore we decided to consider the questionnaire suggested by the UBI><Banca Group[5] and we adopted it with slight modifications.

2.1. The questionnaire

Our questionnaire consists of 26 items that measure different aspects of the latent trait (i.e. the risk profile) and 7 questions that emphasize certain social and financial characteristics: gender, graduate or post-graduate studies, personal or family experiences in the fixed income market, stock market, or with other financial instruments.

The 26 items, as indicated in the questionnaire proposed by UBI><Banca, are divided into three macro areas, which represent three different dimensions in terms of the Rasch conception:

- We have 15 items belonging to the sphere of knowledge, (namely C1, C2, …, C15). The formulation of the items of this dimension is "Could you indicate your level knowledge of the following financial products and instruments?" Government bonds (C1), Deposit certificates (C2), Stocks (C3), Bonds (C4), Implicit derivatives (C5), Structural bonds (C6), Investment funds (C7), SICAV-ETF-ETC (C8), Insurance policies (C9), Policies index linked or unit linked (C10), Certificates (C11), Warrant and covered warrant (C12), Asset management (C13), Hedge funds (C14), Derivatives on OTC (C15).
- We have 6 items directed at investor preferences in liquidity, risk and financial instruments and that consider the growth of investment in the medium-to-long term with limited or strong fluctuations (namely R1, R2,…, R6). In this second dimension we consider four questions as the following way: "How much of your financial asset would you use: in liquidity (R1), for the protection of the capital invested (R2), for the growth of the capital invested in the medium-long term with limited fluctuations (R3), for the growth of the capital invested in the medium-long term with strong fluctuations (R4)?". The fifth question asks how the subject perceives his/her risk profile in relation to the investor behavior (R5). The last question of this sphere concerns the re-evaluation of the financial asset due to an increase of 20% of the financial market (R6).
- We have 5 items that deal with the investor's temporal horizon in order to determine what percentage of the financial assets should be allocated to investments in the very short, short, medium, long, very long term (namely, T1,T2,…,T5). The questions are worded as follows: "How much of your financial asset would you invest in the very short term (T1), short term (T2), medium term (T3), long term (T4), very long term (T5)?"

For each item on knowledge we have considered only three possible response categories: no knowledge (score 0), some knowledge (score 1) and good knowledge (score 2). While for the items that evaluate the percentage of assets to be invested within a given temporal horizon, or with a given risk position, we consider three possible response categories: less than 30% (score 0), between 30% and 70% (score 1), more than 70% (score 2). Finally, there is an item that asks each potential investor to include himself/herself among three possible response categories: strongly risk-averse (score 0), weakly risk-averse (score 1), weakly or strong risk-lover (score 2).

2.2 The sample

The respondents to the questionnaire, described in the previous subsection, were 199 Economics and Business Administration students at Bergamo University. Among these 199 respondents, 48% are male (52% female); 60% follow a degree course, 26% post-graduate studies and 14% other specializations. With regard to personal or family experience in the fixed income and stock markets, we observe that there are generally people in the respondent's family with experience in the fixed income and stock markets, while only few students have direct experience on the markets. Note that experience is prevalent in the fixed income market. In 71% of households, income derives from employment, and, in the remaining 29%, from freelance activities. The annual family net income is less than 30,000 euros for 38% of families, between 30,000 and 60,000 euros for

47% of families and more than 60,000 euros for 15% of families. However, the annual net wealth of families (derived from both financial holdings and real estate) is zero for 11% of families, less than 50,000 euros for 32% of families, between 50,000 and 200,000 euros for another 32% of families, and more than 200,000 euros for 24% of families.

3. Unidimensional approaches to the risk profile questionnaire

All the model within the family of Rasch models are characterized by unidimensionality and additivity. Unidimensionality means that a single construct is being measured (i.e. the latent trait is a scalar). Rasch models produce measurements on an interval scale. This implies additivity on the scale and invariance over the entire continuum, if the data fit the model.

The use of a Rasch models enables predictions of how persons at each level of ability are expected to perform regarding each item. This capability of having estimates for item hierarchy and a person's ability levels enables us to detect "aberrant patterns", such as someone failing to endorse the least severe (or easiest) items while endorsing the most severe (hardest) items. As mentioned above, the simple Rasch model (the dichotomous model) predicts the conditional probability of an ordered binary outcome, given the person's ability and the item's difficulty. This probability is expressed in logistic form. If "correct" answers are coded as 1 and "incorrect" answers are coded as 0, the model expresses the probability of obtaining a correct answer as a function of the size of the difference between the ability of the respondent and the difficulty of the item. The Rasch model may be extended to the polytomous case in several different ways. In our case study we consider the Partial Credit Model (PCM) (Masters, 1982) in which the probability that person *v* responds to item *i* in category *h* is given by:

$$P(h|\vartheta_v,\delta_{ih}) = \frac{\exp\left\{h\vartheta_v - \sum_{j=0}^{h}\delta_{ij}\right\}}{1+\sum_{t=1}^{m}\exp\left\{t\vartheta_v - \sum_{j=1}^{t}\delta_{ij}\right\}} \qquad where \quad h=0,1,...,m$$

where ϑ_v (v = 1, 2, ..., n) is the person parameter, and δ_{ij} (i = 1, 2, ..., k) is an item parameter, also called *step* parameter (for convenience $\delta_{i0}=0$). The step parameter can also be expressed as $\delta_{ij}=\gamma_i+\tau_{ij}$, $j=1,...,m$, where τ_{ij} is the *j-th threshold* parameter of item *i* (for convenience $\tau_{i0}=0$ and $\sum_{j=1}^{m}\tau_{ij}=0$) and γ_i is the item *location* parameter. This formula indicates the probability of a response involves all thresholds of an item. Therefore if a respondent gives a score of 0 (first response category), no threshold is crossed and no threshold appears in the numerator. If the person gives a score of 1 (second response category), only the first threshold is crossed and only the first threshold appears in the numerator. The denominator is the sum of all possible numerators for an item.

Notice that, in a logit form (i.e. the logarithm of the ratio between the probability that the subject responds in category *h* and the probability that the subject responds in category *h* – *1*), we may also write equivalently:

$$\ln\left(\frac{P(h|\vartheta_v,\delta_{ih})}{P(h-1|\vartheta_v,\delta_{ih})}\right)=\vartheta_v-\delta_{ih}\,.$$

The PCM is able to compare respondents and items directly. This means that we have created respondent-free measures and item-free calibrations - abstract measures that transcend specific respondent abilities and specific item difficulties -. This characteristic is sometimes called *specific objectivity*. Thus, the measures represent a respondent's ability as independent of the specific tested items, and item difficulty as independent of a specific sample.

Once the parameters model are estimated, it is interesting to deal with issues of unusual patterns or "misfitting" cases, and thus to compute expected (predicted) response patterns for each person on each item. "Fit statistics" are then derived from a comparison of the expected patterns and the observed patterns. These "fit statistics" are used as a measure of the validity of the data-model fit. The "fit statistics" measure how the observed situation differs from the situation proposed by the theoretical model. In the PCM two groups of "fit statistics" can be considered: one related to subjects and one related to items.

"Person fit" statistics measure the extent to which a person's pattern of responses to the items corresponds to that predicted by the model. A valid response requires that a person of a given ability should have a greater probability of providing a higher rating on easier items than on more difficult items. Therefore if a respondent is more skilled (i.e. he/she is positioned with a higher value of the latent trait) it is expected that he/she will endorse a greater number of items than a subject less skilled.

"Item fit" statistics are used to identify items that may not contribute to a unitary scale or whose response depends on a response to other items. The model requires that an item should have a greater probability of yielding a higher rating for persons with higher ability than for persons with lower ability. Those items identified as not fitting the model need to be examined and revised, eliminated, or possibly calibrated with other misfitting items to determine if a second coherent dimension may exist. There are many potential reasons why an item may misfit. For example, an item may not be related to the rest of the scale or may simply be statistically redundant with reference to the information provided by other items.

As suggested by Wright and Masters (1982), "Item fit" and "Person fit" statistics are based on a standardized comparison between expected and observed scores and they are transformed by the software, used for estimate the scores, into approximate normal deviates. Usually the software also provides a 95% confidence interval for the expected value of these statistics.

Several reasons explain the usefulness of Rasch modeling. To summarize, the advantages of Rasch models include the characteristic of equating responses from different sets of items intended to measure the same construct; the development of equal interval units of measurement when the data fit the model; and the possibility of conducting validity and reliability assessments in one analysis for both item calibration and person measures.
Rasch models also allow for the estimation of person ability freed from the sampling distribution of the items attempted; for the estimation of item difficulty freed from the sampling distribution of the sample employed; and for the expression of item calibration and person measures on a common linear scale (Zhu, Timm, & Ainsworth, 2001).

3.1. The results

Before estimating the PCM calibrations of items and persons, we decided to analyze the correlation matrix between items' raw scores (see Table 1). Looking at this matrix it can be seen that the correlation coefficients are almost all positive with several small values that in some cases are close to zero. In Table 1 we note in bold the coefficients greater than or equal to 0.60: all these coefficients (except one) are in the items belonging to the dimension of knowledge. In this dimension all the values are positive and (except in very few cases) greater than, or equal to, 0.30. The only value greater than 0.60 outside the dimension of knowledge is the ratio between T4 and T5 items concerning investment in the long and very long term.

Table 1. The correlation matrix between items (row scores)

	C1	*C2*	*C3*	*C4*	*C5*	*C6*	*C7*	*C8*	*C9*	*C10*	*C11*	*C12*	*C13*	*C14*	*C15*	*R1*	*R2*	*R3*	*R4*	*R5*	*R6*	*T1*	*T2*	*T3*	*T4*	*T5*
C1	1																									
C2	0.58	1																								
C3	0.55	0.53	1																							
C4	**0.62**	0.47	**0.75**	1																						
C5	0.44	0.52	0.46	0.41	1																					
C6	0.42	0.42	0.43	0.50	**0.70**	1																				
C7	0.56	0.56	**0.60**	**0.60**	0.51	0.47	1																			
C8	0.52	0.46	0.47	0.48	0.56	0.55	0.53	1																		
C9	0.20	0.25	0.32	0.24	0.16	0.23	0.37	0.24	1																	
C10	0.38	0.49	0.47	0.44	0.57	0.52	0.49	0.51	0.47	1																
C11	0.21	0.35	0.27	0.23	0.45	0.41	0.27	0.39	0.24	0.45	1															
C12	0.35	0.36	0.43	0.35	0.52	0.50	0.44	0.57	0.17	0.44	0.52	1														
C13	0.35	0.37	0.43	0.37	0.43	0.45	0.51	0.53	0.47	0.52	0.36	0.45	1													
C14	0.50	0.52	0.58	0.46	0.57	0.53	0.57	0.58	0.31	0.53	0.40	0.55	0.52	1												
C15	0.54	0.43	0.45	0.46	**0.65**	0.59	0.41	0.64	0.08	0.46	0.45	0.55	0.41	**0.64**	1											
R1	0.01	-0.03	-0.03	-0.06	0.01	-0.09	0.00	-0.02	-0.08	0.00	-0.01	-0.02	0.03	0.04	0.05	1										
R2	0.11	0.00	0.02	0.01	0.01	-0.01	0.04	-0.04	-0.04	0.01	-0.02	0.02	-0.01	0.03	0.00	0.08	1									
R3	0.14	0.01	0.07	0.15	0.04	0.08	0.08	0.12	-0.11	0.02	0.04	0.13	0.05	0.10	0.13	0.02	0.34	1								
R4	-0.08	0.01	0.02	-0.01	0.03	0.07	0.04	-0.04	0.10	0.04	0.04	0.01	0.12	0.03	0.01	-0.05	-0.01	0.22	1							
R5	0.17	0.18	0.26	0.24	0.29	0.30	0.26	0.31	0.19	0.18	0.25	0.41	0.35	0.33	0.23	0.01	0.09	0.16	0.32	1						
R6	-0.11	-0.04	0.01	-0.05	0.04	0.04	-0.06	-0.05	0.08	-0.06	0.12	0.08	0.01	-0.04	0.01	-0.11	-0.03	0.04	0.14	0.11	1					
T1	0.00	-0.16	-0.03	0.01	-0.10	-0.05	-0.10	-0.10	0.02	-0.17	0.00	-0.08	-0.06	-0.06	-0.02	0.10	0.13	0.04	0.08	-0.01	0.00	1				
T2	0.14	0.07	0.16	0.11	0.11	0.10	0.04	0.04	0.11	0.05	-0.04	0.05	0.03	0.13	0.15	0.09	0.15	0.16	0.02	0.10	0.17	0.29	1			
T3	0.21	0.16	0.25	0.23	0.14	0.15	0.18	0.19	0.12	0.11	0.15	0.21	0.17	0.24	0.21	0.02	0.21	0.28	0.17	0.32	0.01	-0.06	0.23	1		
T4	0.05	0.05	0.09	0.03	0.13	0.05	0.16	-0.06	-0.01	-0.04	0.00	0.11	0.03	0.10	0.00	-0.08	0.19	0.31	0.30	0.26	0.12	-0.13	0.00	0.41	1	
T5	-0.09	-0.04	0.03	-0.01	0.03	-0.01	0.02	-0.11	-0.10	-0.09	-0.07	-0.08	-0.04	-0.03	-0.05	0.01	0.17	0.18	0.37	0.12	0.08	0.02	-0.06	0.26	**0.66**	1

Note: In bold the correlation coefficients with absolute value ≥ 0.60.

In order to measure the latent variable "predisposition to risk/earn" we apply a preliminary analysis with the PCM model. Thus, parameters have been estimated for each subject and each item using the software package ConQuest (Wu *et al*., 2007).

We consider two different types of unidimensional analysis of the dataset: *i)* a *composite* approach and; *ii)* a *consecutive* approach. In the composite approach we consider a single combined unidimensional calibration of all the items in the questionnaire. In the unidimensional consecutive approach, three separate analyses are performed (one for each dimension): Knowledge (ϑ_1), Risk preferences (ϑ_2) and Time horizon (ϑ_3). As a result of these preliminary analyses we observed that item C9 (knowledge about insurance policies) always presents the worst fit. Hence that item was excluded from the analysis. Table 2 contains the item location estimates for the composite and the consecutive approaches. Table 2 also shows the values of the item-level fit statistics, for every single item. We recall that ConQuest produces the mean squared (MNSQ) fit statistic for every estimated parameter, which is based on a standardized comparison between expected and observed scores. When the model fits the data, MNSQ statistics have a unitary expected value. These statistics are transformed by the software to approximate normal deviates, denoted by *T*. The software also provide a 95% confidence interval (IC) for the expected value of the MNSQ. If the MNSQ fit statistic lies outside IC then the corresponding *T* statistic will have an absolute value that roughly exceeds 2 (see Wu *et al.*, 2007, p. 23).

Overall the fit is not very satisfactory for the composite approach (because several items present too high or too low level of the fit residual statistic), while in the consecutive approach this fit problem is less evident.

Table 2. Comparison between PCM unidimensional analysis (2.A) and PCM consecutive analysis (2.B, 2.C and 2.D) 2.A - PCM unidimensional analysis (all items)

2.A - PCM unidimensional analysis (all items)

Item	Estimate	Error	MNSQ	CI	T
C1	-1.487	0.141	0.83	(0,83, 1,17)	**-2.1**
C2	0.865	0.12	0.91	(0,83, 1,17)	-1
C3	-1.634	0.142	0.79	(0,83, 1,17)	**-2.7**
C4	-1.317	0.135	0.8	(0,83, 1,17)	**-2.4**
C5	1.493	0.129	0.73	(0,78, 1,22)	**-2.7**
C6	1.65	0.135	0.79	(0,80, 1,20)	**-2.2**
C7	0.075	0.116	0.8	(0,83, 1,17)	**-2.4**
C8	1.154	0.119	0.8	(0,80, 1,20)	**-2.1**
C10	1.331	0.127	0.89	(0,81, 1,19)	-1.2
C11	2.388	0.162	0.9	(0,75, 1,25)	-0.7
C12	1.569	0.131	0.8	(0,78, 1,22)	-1.9
C13	0.962	0.126	0.88	(0,82, 1,18)	-1.3
C14	1.181	0.125	0.74	(0,82, 1,18)	**-3.1**
C15	1.063	0.114	0.79	(0,81, 1,19)	**-2.3**
R1	2.149	0.153	1.38	(0,75, 1,25)	2.7
R2	1.418	0.137	1.28	(0,82, 1,18)	2.8
R3	1.559	0.135	1.22	(0,82, 1,18)	2.2
R4	2.879	0.193	1.13	(0,67, 1,33)	0.8
R5	1.026	0.123	1.04	(0,82, 1,18)	0.5
R6	1.645	0.142	1.39	(0,83, 1,17)	4
T1	2.042	0.149	1.42	(0,76, 1,24)	**3.1**
T2	1.94	0.144	1.26	(0,81, 1,19)	**2.5**
T3	1.981	0.147	1.01	(0,84, 1,16)	0.1
T4	1.782	0.141	1.21	(0,80, 1,20)	**2**
T5	1.8	0.14	1.37	(0,77, 1,23)	**2.9**

Note: In bold the t-statistics with absolute value ≥ 2.0

2.B - PCM consecutive analysis: Knowledge

Item	Estimate	Error	MNSQ	CI	T
C1	-2.211	0.171	0.99	(0,81, 1,19)	-0.1
C2	1.318	0.15	1.15	(0,80, 1,20)	1.5
C3	-2.441	0.172	0.82	(0,81, 1,19)	-1.9
C4	-2.009	0.165	0.92	(0,80, 1,20)	-0.8
C5	2.379	0.162	0.83	(0,76, 1,24)	-1.5
C6	2.581	0.168	0.93	(0,78, 1,22)	-0.6
C7	0.088	0.144	0.93	(0,81, 1,19)	-0.8
C8	1.849	0.15	0.91	(0,77, 1,23)	-0.7
C10	2.064	0.158	1.08	(0,79, 1,21)	0.7
C11	3.794	0.197	1.23	(0,75, 1,25)	1.8
C12	2.489	0.164	1.1	(0,76, 1,24)	0.9
C13	1.452	0.156	1.19	(0,80, 1,20)	1.8
C14	1.814	0.156	0.88	(0,80, 1,20)	-1.1
C15	1.713	0.145	1.01	(0,77, 1,23)	0.1

2.C - PCM consecutive analysis: Risk

Item	Estimate	Error	MNSQ	CI	T
R1	1.832	0.145	1.05	(0,75, 1,25)	0.4
R2	1.23	0.128	1	(0,83, 1,17)	0
R3	1.339	0.127	0.99	(0,82, 1,18)	-0.1
R4	2.489	0.185	0.97	(0,66, 1,34)	-0.2
R5	0.864	0.114	0.97	(0,83, 1,17)	-0.3
R6	1.434	0.134	1.05	(0,84, 1,16)	0.6

2.D - PCM consecutive analysis: Time

Item	Estimate	Error	MNSQ	CI	T
T1	1.98	0.148	1.17	(0,76, 1,24)	1.3
T2	1.89	0.143	1.09	(0,81, 1,19)	0.9
T3	1.925	0.146	0.92	(0,84, 1,16)	-1
T4	1.738	0.14	0.91	(0,80, 1,20)	-1
T5	1.752	0.139	0.93	(0,77, 1,23)	-0.6

Clearly, in this consecutive approach, a possible correlation among different dimensions is not taken into account. Thus, we propose to analyze the questionnaire using a three-dimensional Rasch model that also takes the correlation among different dimensions into account.

4. A multidimensional Rasch model

Assuming a multidimensional approach, each respondent v may be measured by a profile of estimates of a three dimensional parameter $\vartheta_v^{\mathrm{T}} = \left(\vartheta_1, \vartheta_2, \vartheta_3\right)$, where the latent traits ϑ_1, ϑ_2 and ϑ_3 are allowed to be correlated. The *Multidimensional Random Coefficients Multinomial Logit Model* (MRCMLM, Adams et al., 1997) – represents the most general structure of a multidimensional Rasch model. For the MRCMLM, the probability of a person *v* scoring *h* on item *i* is given by

$$P(S_{vi} = h \mid \boldsymbol{\theta}_v, \xi) = \frac{\exp(\mathbf{b}_{ih}^{\mathrm{T}} \boldsymbol{\theta}_v + \mathbf{a}_{ih}^{\mathrm{T}} \xi)}{\sum_{j=0}^{m_i} \exp(\mathbf{b}_{ij}^{\mathrm{T}} \boldsymbol{\theta}_v + \mathbf{a}_{ij}^{\mathrm{T}} \xi)}$$

where:

$m_i + 1$ is the number of ordered response categories in item *i*;

S_{vi} represents the response "score", of person *v* on item *i*, with values 0, 1, …, m_i

D is the number of hypothesized dimensions for the latent trait vector;

$\vartheta_v^{\mathrm{T}} = \left(\vartheta_{v1}, \vartheta_{v2}, \ldots, \vartheta_{vD}\right)$ is the latent trait vector for person *v*;

ξ is the vector of item parameters;

$\mathbf{b}_{ih}^{\mathrm{T}} = \left(\mathrm{b}_{ih1}, \mathrm{b}_{ih2}, \ldots, \mathrm{b}_{ihD}\right)$ is a scoring weights vector of known constants,

b_{ihd} represents the score given to category *h* on dimension *d* of item *i* (by definition, the score for a response in the 0 category is 0 for both dimensions);

$\mathbf{a}_{ih}^{\mathrm{T}}$ is a design vector of known constants given to category *h* of item *i*;

The design vectors and the scoring weight vectors can be collected to form, respectively, a design matrix, **A**, and a scoring weight matrix, **B**. Within the Rasch family, the desired model is obtained by suitable choice of matrices **A** and **B**. In particular, as a special case of a MRCMLM (with *D* equal to 1, and for convenient choices of **A** and **B**), we may obtain the PCM. In this paper the MRCML is estimated by using the program ConQuest (Wu *et al*., 2007). ConQuest software provides Marginal Maximum Likelihood (MML) estimates of the parameters of the model – by adopting Bock and Aitkin's (Bock & Aitkin, 1981) formulation of the Expectation-Maximization algorithm (Dempster *et al*., 1977). The MML approach assumes that persons have ϑ vectors that are sampled from a population in which the distribution of ϑ is given by a multivariate density function $g\left(\vartheta;\omega\right)$, where ω indicates a vector of parameters that characterize the distribution. In MML estimation, the vector of item parameters is simultaneously estimated with the parameter ω of the latent trait distribution by maximizing the marginal likelihood function

$$L_M = \prod_{v=1}^{n} \int_{\theta} p\left(s_v, \vartheta, \delta, \omega\right) d\vartheta \ ,$$

where $p\left(s_v, \vartheta, \delta, \omega\right) = \prod_{i=1}^{k} P(S_{vi} = s_{vi} \mid \vartheta, \delta) g\left(\vartheta, \omega\right)$ represents the simultaneous distribution of the response pattern s_v and the ability ϑ for any person *v* in the sample, and where δ represents the vector of *item x category* parameters (also known as step difficulties) that have been gathered into the vector ξ. Now, if *g* is constrained as a normal distribution, with mean set equal to zero to identify the model, then $\omega = (\mathbf{0}, \Sigma)$, where the covariance matrix $\Sigma = (\sigma_{ij})$ is an estimand. In particular, the MML approach allows the correlations between latent variables to be estimated *directly*, avoiding the problems associated with the influence of measurement error on computing correlations by using the estimated parameters.

In this case study we should define a MRCMLM with *D*=3 and $m_i + 1 = 3$ for every *i*, $i = 1,...,k$. The same dataset is analysed with two models, say *M1* and M2. These models are instances of two different types of multidimensionality. Model *M1* is termed between-item multidimensional. Indeed, in such a model each item is related to only one particular latent trait. Model *M2* is an instance of a within-item multidimensional model. We have a within-item multidimensional model when the responses to some (i.e., at least one) of the items depend on more than one dimension (i.e., require "abilities" from more than one latent trait). In particular, Model *M2* assumes that responses of items from R1 through R4 depend on both the dimensions ϑ_1 and ϑ_2. This feature makes the model *compensatory* – as regards the items from R1 through R4; this means that we assume that it is possible for a respondent with a low level on one latent trait to compensate for this by having a high level on another latent trait. Table 3 shows explicitly - for both the models *M1* and *M2* - the probabilities of observing the scores 0, 1 and 2, for each of the items considered. Figures 1 and 2 illustrate graphically the hypothesized structure of these two models.

Table 3. Probability functions for models M1 and M2

Items	Score	Probability	
		Model *M1*	Model *M2*
C1-C14	0	c_1^{-1}	c_1^{-1}
	1	$c_1^{-1}\exp(\theta_{v1} - \delta_{i1})$	$c_1^{-1}\exp(\theta_{v1} - \delta_{i1})$
	2	$c_1^{-1}\exp(2\theta_{v1} - \delta_{i1} - \delta_{i2})$	$c_1^{-1}\exp(2\theta_{v1} - \delta_{i1} - \delta_{i2})$
R1-R4	0	c_2^{-1}	c_2^{-1}
	1	$c_2^{-1}\exp(\theta_{v2} - \delta_{i1})$	$c_2^{-1}\exp(\theta_{v1} + \theta_{v2} - \delta_{i1})$
	2	$c_2^{-1}\exp(2\theta_{v2} - \delta_{i1} - \delta_{i2})$	$c_2^{-1}\exp(2\theta_{v1} + 2\theta_{v2} - \delta_{i1} - \delta_{i2})$
R5-R6	0	c_2^{-1}	c_3^{-1}
	1	$c_2^{-1}\exp(\theta_{v2} - \delta_{i1})$	$c_3^{-1}\exp(\theta_{v2} - \delta_{i1})$
	2	$c_2^{-1}\exp(2\theta_{v2} - \delta_{i1} - \delta_{i2})$	$c_3^{-1}\exp(2\theta_{v2} - \delta_{i1} - \delta_{i2})$
T1-T5	0	c_3^{-1}	c_4^{-1}
	1	$c_3^{-1}\exp(\theta_{v3} - \delta_{i1})$	$c_4^{-1}\exp(\theta_{v3} - \delta_{i1})$
	2	$c_3^{-1}\exp(2\theta_{v3} - \delta_{i1} - \delta_{i2})$	$c_4^{-1}\exp(2\theta_{v3} - \delta_{i1} - \delta_{i2})$

Note: The constant c represents the sum of the numerators – within the formula of the probability function.

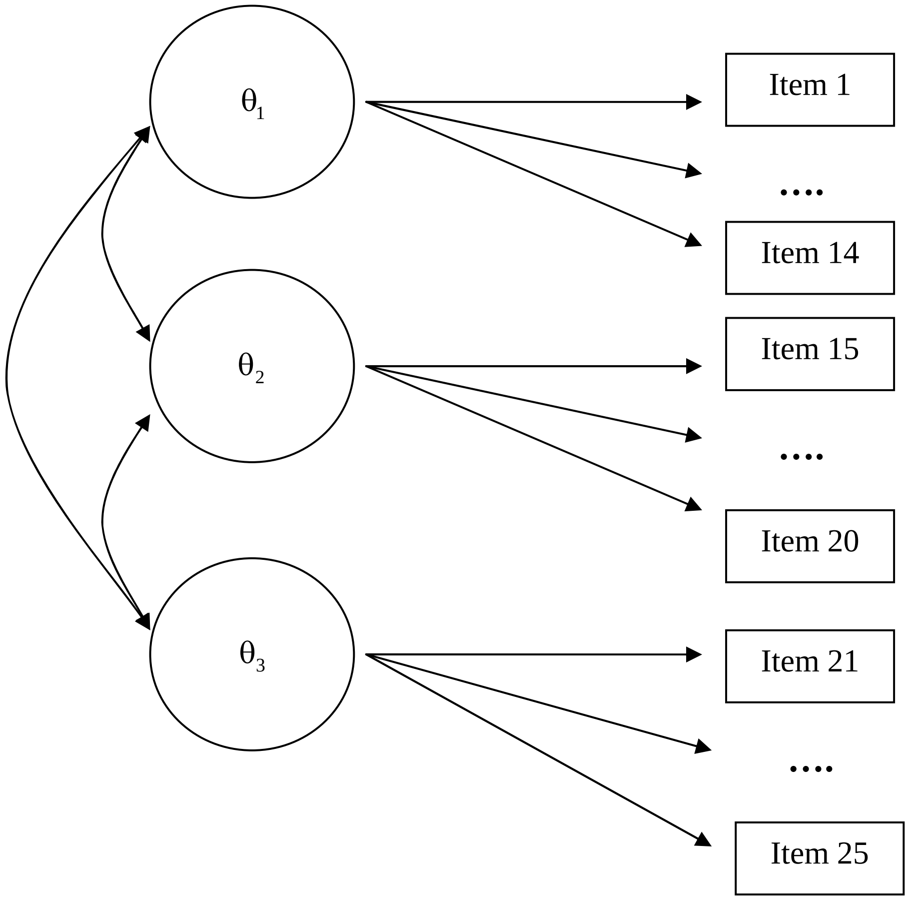

Figure 1. Conceptual path diagram of Model M1

Model-selection criteria typically take the form of a penalized likelihood function, that is the deviance *d* plus a penalty term, which increase with the number *p* of parameters. For example, the AIC index (Akaike, 1987) is defined as $\mathrm{AIC}(model) = d + 2p$. The Schwarz's BIC index (Schwarz, 1978) takes the form $BIC(model) = d + p \cdot (\log N)$, where *N* is the sample size. The model associated with the smallest value of AIC (or BIC) is considered the best fitting model. In our case, since the number of parameters to be estimated is the same (*p*=56) for both the models *M1* and *M2*, the deviance may be used to compare statistically the fit of these two models. Comparing the deviances of these models, we find the values 6775.74 and 6786.80, for *M1* and *M2* respectively. Then it can be concluded that the fit of model *M2* is worse than the fit of the model *M1*. Moreover, as Table 4 indicates, on the basis of a comparison of the AIC, as well as the BIC, we can state that the multidimensional models fit the data better than the both the composite and the consecutive models.

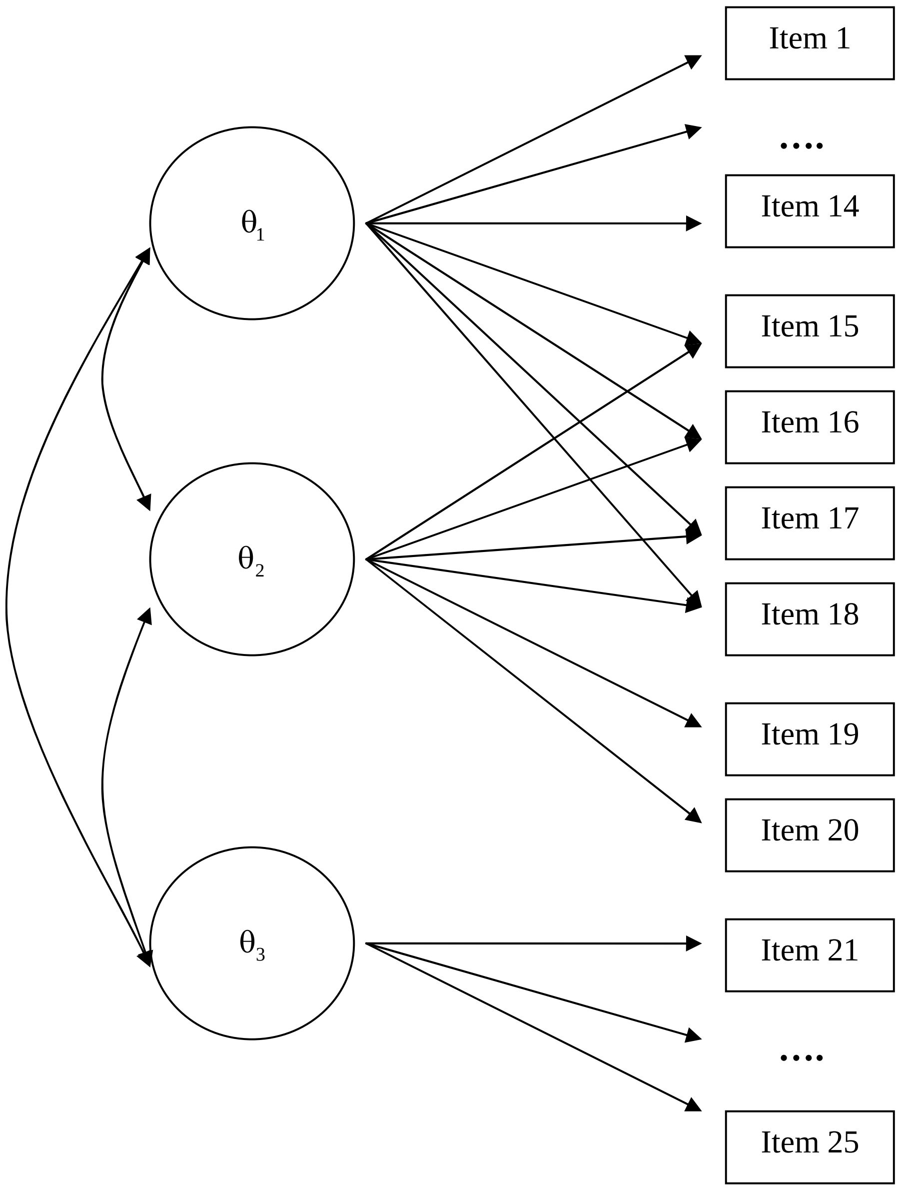

Figure 2. Conceptual path diagram of Model M2

Table 4. Comparing unidimensional and multidimensional models

MODEL	*# of estimated parameters*	*AIC*	*BIC*
Unidimensional composite	*51*	*7463,92*	*7631,62*
Unidimensional consecutive (3 analyses combined)	*53*	*6937,33*	*7111,61*
Multidimensional M1	*56*	*6887,74*	*7071,88*
Multidimensional M2	*56*	*6898,80*	*7082,94*

Now, running the analysis for model *M1*, ConQuest provides the estimates of the item parameters and the covariance matrix. As mentioned above, for each of these estimates ConQuest also gives an estimate of the (asymptotic) standard error and a diagnostic index of fit, *T*. The item parameter estimates for model *M1* are reported in Table 5. Notice that the item parameter estimate (item location) can be viewed as the average of the step parameters, i.e. $(\delta_1 + \delta_2)/2$. As can be seen, compared with those given by the previous analysis with the unidimensional composite model, all the fit statistics (column *T*) now look good. As a general rule of interpretation, one should take into account that high positive *T* values (e.g. above 2) denote unmodelled noise, while high negative *T* values (e.g. below -2) denote unmodelled dependencies in the data.

Table 5. The item parameter estimates - Multidimensional model M1

	item	estimate	error	MNSQ	CI	T
C1	1	-2.203	0.169	0.96	(0.81, 1.19)	-0.4
C2	2	1.262	0.149	1.18	(0.80, 1.20)	1.7
C3	3	-2.423	0.17	0.85	(0.81, 1.19)	-1.6
C4	4	-2.016	0.163	0.89	(0.81, 1.19)	-1.1
C5	5	2.312	0.161	0.86	(0.76, 1.24)	-1.1
C6	6	2.517	0.167	0.92	(0.78, 1.22)	-0.7
C7	7	0.042	0.143	0.93	(0.81, 1.19)	-0.7
C8	8	1.783	0.15	0.88	(0.77, 1.23)	-1
C10	9	2.003	0.157	1.09	(0.79, 1.21)	0.9
C11	10	3.719	0.196	1.24	(0.76, 1.24)	1.9
C12	11	2.423	0.164	1.1	(0.76, 1.24)	0.8
C13	12	1.398	0.156	1.21	(0.80, 1.20)	2
C14	13	1.755	0.155	0.86	(0.80, 1.20)	-1.4
C15	14	1.645	0.144	0.97	(0.77, 1.23)	-0.2
R1	15	1.946	0.148	1.15	(0.75, 1.25)	1.2
R2	16	1.297	0.132	0.99	(0.83, 1.17)	-0.1
R3	17	1.416	0.13	0.97	(0.82, 1.18)	-0.3
R4	18	2.635	0.189	0.96	(0.66, 1.34)	-0.2
R5	19	0.922	0.117	0.98	(0.83, 1.17)	-0.2
R6	20	1.51	0.137	1.1	(0.84, 1.16)	1.2
T1	21	1.974	0.146	1.15	(0.76, 1.24)	1.2
T2	22	1.89	0.141	1.06	(0.81, 1.19)	0.7
T3	23	1.936	0.145	0.96	(0.84, 1.16)	-0.5
T4	24	1.735	0.138	0.9	(0.80, 1.20)	-1
T5	25	1.747	0.137	0.93	(0.77, 1.23)	-0.6

Table 6 reports the correlations between each of the three latent traits ϑ_1, ϑ_2 and ϑ_3, under the multidimensional model *M1*. Obviously, the estimated correlation is not the same as the correlation of the estimates. Table 7 shows the correlations of the person parameter estimates (expected a posteriori) obtained from the consecutive approach. As to be expected, under this latter approach the correlations between the latent traits result underestimated.

Table 6. Correlations between the latent variables ϑ_1, ϑ_2 and ϑ_3 - Multidimensional model

ϑ_1	1		
ϑ_2	0.332	1	
ϑ_3	1.166	0.910	1
Var	4.560	0.568	0.843

Table 7. Correlations between the latent variables ϑ_1, ϑ_2 and ϑ_3- Consecutive approach

ϑ_1	1		
ϑ_2	0.100	1	
ϑ_3	0.024	0.331	1

Concluding remarks

Given the multivariate position of each investor with respect to the three latent traits (personal knowledge, risk predisposition and temporal horizon) we can represent his/her position in the possible investments proposed by the bank. This can be done considering either a uni-dimensional consecutive approach where three separate analyses are performed for each dimension or considering a three-dimensional approach that accounts the common relationships among the latent variables. The proposed empirical analysis shows that the three-dimensional approach is much more appealing by a statistical point of view. Thus, for this sample, the second approach appears the one that should be used to parameterize the individual choices.

The empirical results can be used to opportunely rescale the three dimensions so that we can prospect different situations that characterize the investors' choices. In particular, for each investor, we can describe in a tri-dimensional space the percentage that should be invested in some typical financial instruments (contingent claims, stocks, bonds, treasury bills) considering their characterization with respect to the latent traits. This first pre-selection should be further improved in a more detailed portfolio selection that account for the personal risk tolerance and temporal horizon according to the utility theory under uncertainty conditions. In this context the proposed analysis represents an alternative methodology of choice for the portfolio selection problem.

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