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MONTE CARLO ANALYSIS OF CHANGE POINT ESTIMATORS

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Abstract: *We consider several estimators for the change point in a sequence of independent observations. These are defined as the maximizing points of usually used statistics for nonparametric change point detection problems. Our investigations focus on the non asymptotic behaviour of the proposed estimators for sample sizes commonly observed in practice. We conducted a broad Monte Carlo study to compare these change point estimators, also investigating their properties and potential* **practical applications.**

Key words: *change point estimation; nonparametric tests; stochastically ordered alternatives;* Mann-Whitney statistic

1. Introduction

Since the middle of the twentieth century, the retrospective change point problem has been extensively addressed in statistics and engineering literature (e.g., Chernoff and Zacks, 1964; James et al., 1987; Csorgo and Horvath, 1988; Gombay and Horvath, 1994, Gurevich and Vexler, 2005, 2010; Gurevich, 2006, 2007, 2009). This problem is directly connected to process capability and is important in biostatistics, education, economics and other fields (e.g., see Page, 1954, 1955; Sen and Srivastava, 1975). Formally, the problem is that of hypothesis testing:

H_0, the null: $X_1, X_2, ..., X_n \sim F_1$ versus (1)

H_1, the alternative: $X_i \sim F_1$, $X_j \sim F_2$, $i = 1,...,\nu-1$, $j = \nu,...,n$,

where $X_1, X_2, ..., X_n$ is a sample of independent observations, F_1 and F_2 are distribution functions with corresponding density functions f_1 and f_2, respectively. The distribution functions F_1 and F_2 are not necessary known. The unknown parameter ν, $2 \le \nu \le n$ is called a change point. In accordance with the statistical literature, the problem (1) has been investigated in parametric and nonparametric forms, depending on assumptions made on the distribution functions F_1 and F_2. In the parametric case of (1), it is assumed that the distribution functions F_1 and F_2 have known forms that can contain certain unknown parameters (e.g., James et al., 1987; Gombay and Horvath, 1994; Gurevich, 2007). In the nonparametric case of (1), the functions F_1, F_2 are assumed to be completely unknown (e.g., Wolfe and Schechtman, 1984; Ferger, 1994; Gombay, 2000, 2001; Gurevich, 2006).

The parametric case of testing the change point problem (1) has been dealt with extensively in both the theoretical and applied literature (e.g., Chernoff and Zacks, 1964; Kander and Zacks, 1966; Sen and Srivastava, 1975; James et al., 1987; Gombay and Horvath, 1994; Gurevich and Vexler, 2005; Gurevich, 2007). Chernoff and Zacks (1964) considered the problem (1) based on normally distributed observations with $F_1 = N(\theta_0, 1)$, $F_2 = N(\theta, 1)$, where θ_0 and $\theta > \theta_0$ are unknown. Kander and Zacks (1966) extended the Chernoff and Zacks's results to a case based on data from the one-parameter exponential family. Sen and Srivastava (1975) used the maximum likelihood technique to present a test-statistic. James et al. (1987) proposed, in the context of (1), decision rules based on likelihood ratios and recursive residuals. This change point literature concluded that there is no a globally (with respect to values of ν, under H_1) preferable test for (1). It turned out that the Chernoff and Zacks' test has a larger power than that of tests based on the likelihood ratio or recursive residuals when ν is around $n/2$, but this property is reversed if the change point ν is close to the edges, i.e., when $\nu \approx n$ or $\nu \approx 2$.

When the problem (1) is stated nonparametrically, the common components of change point detection policies have been proposed to be based on signs and/or ranks and/or U -statistics (e.g., Wolfe and Schechtman, 1984; Ferger, 1994; Gombay, 2000, 2001; Gurevich, 2006). Sen and Srivastava (1975) focused on the problem (1) with the unknown distributions $F_1(x)$, $F_2(x) = F_1(x-\beta)$, $\beta > 0$. The authors suggested to reject H_0, for large values of the statistic

$$D = \max_{2 \le k \le n} \left\{ \left[U_{k-1,n-k+1} - (k-1)(n-k+1)/2 \right] / \left[(k-1)(n-k+1)(n+1)/12 \right]^{\frac{1}{2}} \right\}, \qquad (2)$$

where $U_{k-1,n-k+1} = \sum_{i=1}^{k-1} \sum_{j=k}^{n} I(X_i \le X_j)$, ($I(\cdot)$ is the indicator function), is the Mann-Whitney statistic for two samples of size $k-1$ and $n-k+1$. (Sen and Srivastava (1975) did not study analytical properties of the statistic (2).) Setting the problem (1) in a similar manner to Sen and Srivastava (1975), Pettitt (1979) used the statistic

$$\max_{2 \le k \le n} \left\{ -\sum_{i=1}^{k-1} \sum_{j=k}^{n} Q_{ij} \right\}, \quad Q_{ij} = sign(X_i - X_j) = \begin{cases} 1 & X_i > X_j \\ 0 & X_i = X_j \\ -1 & X_i < X_j \end{cases}$$

to propose a change point detection policy. Wolfe and Schechtman (1984) showed that this statistic can be presented as

$$K = 2\max_{2\le k\le n}\left\{U_{k-1,n-k+1} - (k-1)(n-k+1)/2\right\}. \qquad (3)$$

Csorgo and Horvath (1988) have modified very slightly the statistic (2) and evaluated asymptotically $(n \to \infty)$ the type I error of the corresponding test. Ferger (1994) and Gombay (2001,2002) studied the asymptotic behaviour of U-statistics, in particular, the asymptotic properties of the test based on statistic (3). Wolfe and Schechtman (1984), Gurevich (2006, 2009) as well as Gurevich and Vexler (2010) compared the powers of various nonparametric retrospective tests for the problem (1). Their study confirmed that the tests based on statistics (2) and (3) are usually very efficient, especially for stochastically ordered alternatives. Moreover, It turned out that, as in the parametric case, there is no a globally preferable test. For $\nu \approx n/2$ it seems that the test based on the statistic (3) is preferable; and for ν that is close to edges, the test based on the statistic (2) is preferable. Note that, under H_0 the distribution of the statistics (2) and (3) does not depend on the distribution of the observations. That is, the tests based on these statistics are exact and corresponding critical values can be tabulated for fix sample sizes and any desirable significance level. When the two-sided alternative $F_2(x) = F_1(x-\beta)$, $\beta \ne 0$ is assumed, the absolute values under the operator max in the statistics (2) and (3) should be considered (e.g., Gurevich and Vexler, 2010). Thus, the tests for the two-sided alternative are based on the statistics

$$DD = \max_{2\le k\le n}\left|U_{k-1,n-k+1} - (k-1)(n-k+1)/2\right| / \left[(k-1)(n-k+1)(n+1)/12\right]^{\frac{1}{2}}, \qquad (4)$$

$$KK = 2\max_{2\le k\le n}\left|U_{k-1,n-k+1} - (k-1)(n-k+1)/2\right|. \qquad (5)$$

While the change point literature mainly relates on testing the hypotheses (1), rather scant work has been done on the problem of estimation of the change point ν. Gurevich and Vexler (2005, 2010) showed that, in general, a process of estimation of the change point parameter ν should be started if needed, provided that just the null hypothesis of (1) is rejected. When H_0 is rejected, the issue to estimate the unknown parameter ν can be stated. Borovkov (1999) as well as Gurevich and Vexler (2005) investigated different estimators of the change point parameter ν in a parametric framework. Ferger (2001) studied the behaviour of change point estimators in a nonparametric framework under the null hypothesis. Theoretical investigations of the change point estimators need substantial mathematical details and usually are restricted to the asymptotic analysis when $\nu \to \infty$, $n-\nu \to \infty$ (e.g. Ferger, 2001 and his references).

In this article we propose four nonparametric change point estimators in the context of the problem (1) with stochastically ordered alternatives. That is, we consider the problem (1), where the functions F_1 and F_2 are completely unknown but the observations after the change are assumed to be stochastically larger/smaller than that before the change. We focus on the non asymptotic behaviour of the proposed estimators and present a broad Monte Carlo study investigating their properties and potential ***practical applications***. The rest of the paper is organized as follows. Section 2 gives a short background in change point estimation and presents four proposed estimators of ν. Section 3 displays a Monte Carlo study. Finally, we state our conclusions in Section 4.

2. The proposed change point estimators

Let $X_1, X_2, ..., X_n$ be independent random observations. We consider the hypotheses (1) where unknown distribution functions F_1 and F_2 such that for all x, $F_2(x) \le F_1(x)$ or $F_2(x) \ge F_1(x)$ (that is, we assume after a possible change the observations are stochastically larger or smaller than before the change). The analysis of change point estimators in a nonparametric framework has been of increasing interest in the last two decades. Commonly, the results for estimators of ν are concerned with the case of an actual change ($2 \le \nu \le n$) and are based on ***theoretical*** studies regarding the ***asymptotic*** ($\nu \to \infty$, $n - \nu \to \infty$) behaviour of their distributions. Since in many actual applications the most commonly used sample sizes are small or average, the practical implementation of such results is not straightforward. Gurevich and Vexler (2005) presented some Monte Carlo experiments regarding the non asymptotic behavior of the maximum likelihood change point estimators in the parametric framework, i.e., when the problem (1) is stated in the parametric form.

Here we propose the following four estimators of ν in a nonparametric framework. These maximum likelihood type estimators are based on the relevant nonparametric statistics D, K, DD, KK that have been suggested for corresponding change point detection problems.

$$\hat{\nu}_D = \arg\max_{2 \le k \le n} \left\{ \left[U_{k-1,n-k+1} - (k-1)(n-k+1)/2 \right] / \left[(k-1)(n-k+1)(n+1)/12 \right]^{\frac{1}{2}} \right\}, \quad (6)$$

$$\hat{\nu}_K = \arg\max_{2 \le k \le n} \left\{ U_{k-1,n-k+1} - (k-1)(n-k+1)/2 \right\}, \quad (7)$$

$$\hat{\nu}_{DD} = \arg\max_{2 \le k \le n} \left| U_{k-1,n-k+1} - (k-1)(n-k+1)/2 \right| / \left[(k-1)(n-k+1)(n+1)/12 \right]^{\frac{1}{2}}, \quad (8)$$

$$\hat{\nu}_{KK} = \arg\max_{2 \le k \le n} \left| U_{k-1,n-k+1} - (k-1)(n-k+1)/2 \right|. \quad (9)$$

Theoretical evaluations of distribution of the proposed estimators ***require complicated computations*** that are beyond the scope of this article. In Section 3 we present Monte Carlo results to illustrate the non asymptotic behaviour of the estimators $\hat{\nu}_D$, $\hat{\nu}_K$, $\hat{\nu}_{DD}$, $\hat{\nu}_{KK}$ as well as their comparisons and the practical suitability.

3. Monte Carlo Study

To study the behavior of the change point estimators (6)-(9), we conducted the following experiments. For each distribution set with different sample sizes, we generated 50,000 times corresponding data. Tables 1;2;3 presents the Monte Carlo means and standard deviations of the estimators $\hat{\nu}_D$, $\hat{\nu}_K$, $\hat{\nu}_{DD}$, $\hat{\nu}_{KK}$, when samples of X's were drawn from $F_1 = Norm(\mu, \sigma^2)$, $F_2 = Norm(\mu + \beta\sigma, \sigma^2)$; $F_1 = \mathrm{Unif}(0,1)$, $F_2 = \mathrm{Unif}(\beta, 1+\beta)$; $F_1 = LogNorm(\mu, \sigma^2)$, $F_2 = LogNorm(\mu + \beta\sigma, \sigma^2)$, respectively, for different values of β, sample sizes n and values of the change point parameter ν. To explain the results of these experiments we have also evaluated the simulated powers of the tests RD, RK,

RDD, RKK, based on the statistics (2), (3), (4), (5). That is, the test RD rejects the null hypothesis H_0 if $D > C_D$, the tests RK, RDD, RKK reject H_0 if $K > C_K$, $DD > C_{DD}$, $KK > C_{KK}$, respectively, where C_D, C_K, C_{DD}, C_{KK} are test thresholds. (The tests RD and RK are proposed for the situations where the observations after the possible change are suspected to be stochastically larger than that before the change. For the situations where the observations after the possible change are suspected to be stochastically smaller, the similar tests that reject H_0 for small values of the statistics D and K should be considered.) The Monte Carlo powers of the tests RD, RK, RDD, RKK were evaluated at the level of significance 0.05 that was fixed experimentally by choosing special values of the thresholds C_D, C_K, C_{DD}, C_{KK}. (Under the null hypothesis of (1), the baseline distribution functions of the nonparametric test statistics D, K, DD, KK do not depend on data distributions, only tables of critical values of the tests are required for their implementation.) Note also that distributions of all four considered statistics do not depend on the parameters μ and σ. Therefore, without loss of generality we utilized $\mu = 0$ and $\sigma = 1$. The power functions of the tests RD, RK, RDD, RKK as functions of ν are symmetric around the middle points $\nu - 1 = n/2$. Moreover, the powers of the tests RDD and RKK do not depend on a sign of β, for all fixed ν.

JOURNAL OF APPLIED QUANTITATIVE METHODS

Table1. The Monte Carlo powers (at $\alpha = 0.05$) of the tests RD, RK, RDD, RKK, and the Monte Carlo means and standard deviations (Std) of the estimators $\hat{\nu}_D$, $\hat{\nu}_K$, $\hat{\nu}_{DD}$, $\hat{\nu}_{KK}$, when $F_1 = Norm(0,1)$, $F_2 = Norm(\beta,1)$, for different sample sizes n and values of β and ν. Observations with the subscript $\nu - 1$ are the last observations before the change.

n	β	$\nu - 1$	Power of RD	Power of RK	Power of RDD	Power of RKK	Mean (Std) of $\hat{\nu}_D - 1$	Mean (Std) of $\hat{\nu}_K - 1$	Mean (Std) of $\hat{\nu}_{DD} - 1$	Mean (Std) of $\hat{\nu}_{KK} - 1$
	0.8	10	0.388	0.429	0.264	0.311	10.0(4.3)	9.9(3.0)	9.9(4.6)	9.9(3.0)
		5	0.291	0.258	0.177	0.160	7.5(4.9)	7.7(3.9)	8.1(5.3)	7.8(3.9)
		3	0.186	0.135	0.105	0.079	7.0(5.6)	7.6(4.6)	8.2(5.9)	7.6(4.6)
	1	10	0.526	0.574	0.387	0.450	10.0(3.6)	9.9(2.5)	10.0(3.8)	9.9(2.5)
20		5	0.396	0.354	0.262	0.227	7.0(4.4)	7.4(3.4)	7.5(4.8)	7.4(3.4)
		3	0.250	0.160	0.141	0.093	6.4(5.2)	7.1(4.3)	7.4(5.7)	7.1(4.3)
	1.2	10	0.661	0.717	0.524	0.603	10.0(3.0)	9.9(2.0)	10.0(3.2)	9.6(2.0)
		5	0.513	0.458	0.359	0.312	6.6(3.8)	7.1(3.1)	7.0(4.1)	7.1(3.0)
		3	0.326	0.194	0.189	0.111	5.9(4.8)	6.8(4.1)	6.7(5.3)	6.8(4.1)
	0.8	20	0.617	0.689	0.497	0.575	20.0(7.1)	19.9(4.3)	20.0(7.5)	19.9(4.4)
		10	0.485	0.461	0.368	0.325	13.4(8.6)	14.4(6.4)	14.2(9.4)	14.4(6.4)
		5	0.284	0.172	0.181	0.101	11.9(11.1)	14.1(9.0)	14.0(12.1)	14.0(9.0)
	1	20	0.798	0.850	0.704	0.770	20.0(5.4)	20.0(3.3)	20.0(5.6)	20.0(3.3)
40		10	0.660	0.627	0.539	0.491	12.4(6.9)	13.6(5.4)	12.7(7.5)	13.6(5.4)
		5	0.396	0.224	0.265	0.132	10.3(9.7)	13.0(8.3)	11.9(10.9)	13.1(8.3)
	1.2	20	0.916	0.944	0.856	0.902	20.0(4.0)	20.0(2.6)	20.0(4.2)	20.0(2.6)
		10	0.806	0.785	0.708	0.657	11.6(5.4)	13.0(4.6)	11.8(5.8)	13.0(4.6)
		5	0.526	0.281	0.374	0.163	9.1(8.4)	12.2(7.6)	10.2(9.4)	12.1(7.7)
	0.8	35	0.838	0.897	0.753	0.834	35.0(9.3)	35.0(5.4)	35.0(9.7)	35.0(5.4)
		50	0.749	0.778	0.645	0.665	47.2(11.1)	45.4(8.0)	46.9(11.8)	45.3(8.0)
		60	0.507	0.357	0.378	0.225	52.7(15.9)	47.9(13.3)	50.6(17.9)	47.9(13.3)
	1	35	0.956	0.979	0.922	0.956	35.0(6.3)	35.0(3.9)	35.0(6.4)	35.0(3.9)
70		50	0.903	0.924	0.849	0.861	48.4(7.7)	46.4(6.3)	48.2(8.0)	46.4(6.3)
		60	0.689	0.514	0.565	0.346	55.3(12.4)	49.9(11.7)	54.3(14.0)	49.9(11.7)
	1.2	35	0.993	0.997	0.985	0.993	35.0(4.4)	35.0(2.9)	35.0(4.3)	35.0(2.9)
		50	0.978	0.982	0.953	0.960	49.0(5.3)	47.2(5.0)	49.0(5.3)	47.2(5.1)
		60	0.838	0.673	0.741	0.490	57.0(9.3)	51.4(10.4)	56.6(10.4)	51.5(10.4)
	0.8	50	0.939	0.971	0.898	0.944	50.0(10.2)	50.0(6.0)	50.1(10.4)	49.9(6.1)
		70	0.889	0.916	0.825	0.857	67.8(11.9)	65.1(9.1)	67.7(12.4)	65.2(9.0)
		90	0.502	0.270	0.374	0.164	79.1(23.4)	69.5(20.9)	75.6(26.9)	69.3(20.8)
	1	50	0.993	0.998	0.985	0.994	50.0(6.5)	50.0(4.3)	50.0(6.6)	50.0(4.2)
100		70	0.979	0.986	0.960	0.969	68.9(7.5)	66.4(6.8)	68.9(7.7)	66.4(6.8)
		90	0.688	0.386	0.564	0.232	83.3(18.0)	72.3(18.8)	81.5(20.8)	72.3(18.8)
	1.2	50	1.000	1.000	0.999	1.000	50.0(4.2)	50.0(3.1)	50.0(4.3)	50.0(3.1)
		70	0.997	0.999	0.995	0.996	69.3(4.9)	67.2(5.3)	69.3(4.9)	67.3(5.4)
		90	0.842	0.521	0.751	0.338	85.8(13.2)	74.6(17.2)	85.1(15.0)	74.6(17.1)
	0.8	75	0.990	0.997	0.981	0.993	74.9(10.3)	75.0(6.6)	75.0(10.5)	75.0(6.7)
		100	0.980	0.990	0.965	0.987	98.8(11.4)	95.9(9.3)	98.8(11.6)	95.8(9.2)
		125	0.875	0.805	0.812	0.682	120.2(18.9)	109.9(19.5)	119.5(20.4)	110.2(19.5)
	1	75	1.000	1.000	0.999	1.000	75.0(6.2)	75.0(4.5)	75.0(6.2)	75.0(4.5)
150		100	0.999	1.000	0.998	0.999	99.4(6.8)	97.0(6.6)	99.3(6.8)	97.0(6.7)
		125	0.974	0.947	0.954	0.895	122.7(11.0)	113.3(15.9)	122.6(11.6)	113.2(16.0)
	1.2	75	1.000	1.000	1.000	1.000	75.0(4.1)	75.0(3.2)	75.0(4.1)	75.0(3.2)
		100	1.000	1.000	1.000	1.000	99.6(4.3)	97.8(5.0)	99.6(4.2)	97.7(4.9)
		125	0.997	0.992	0.994	0.978	123.8(6.5)	115.6(13.4)	123.7(6.7)	115.7(13.4)

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Table2. The Monte Carlo powers (at $\alpha = 0.05$) of the tests RD, RK, RDD, RKK, and the Monte Carlo means and standard deviations (Std) of the estimators $\hat{\nu}_D$, $\hat{\nu}_K$, $\hat{\nu}_{DD}$, $\hat{\nu}_{KK}$, when $F_1 = \mathrm{Unif}(0,1)$, $F_2 = \mathrm{Unif}(\beta, 1+\beta)$, for different sample sizes n and values of β and ν. Observations with the subscript $\nu - 1$ are the last observations before the change.

n	β	$\nu - 1$	Power of RD	Power of RK	Power of RDD	Power of RKK	Mean (Std) of $\hat{\nu}_D - 1$	Mean (Std) of $\hat{\nu}_K - 1$	Mean (Std) of $\hat{\nu}_{DD} - 1$	Mean (Std) of $\hat{\nu}_{KK} - 1$
	0.20	10	0.299	0.329	0.191	0.224	10.0(4.8)	9.9(3.4)	10.0(5.0)	9.9(3.4)
		5	0.225	0.201	0.137	0.121	7.9(5.3)	8.1(4.2)	8.7(5.6)	8.1(4.2)
		3	0.156	0.115	0.091	0.069	7.5(5.9)	8.0(4.8)	8.8(6.1)	8.0(4.8)
	0.35	10	0.627	0.682	0.486	0.555	10.0(3.3)	9.9(2.2)	10.0(3.4)	9.9(2.2)
20		5	0.470	0.419	0.319	0.277	6.8(4.0)	7.2(3.2)	7.2(4.4)	7.2(3.2)
		3	0.294	0.184	0.182	0.105	6.1(5.0)	6.9(4.1)	7.0(5.4)	6.9(4.2)
	0.50	10	0.893	0.924	0.809	0.864	10.0(2.0)	10.0(1.3)	10.0(2.0)	10.0(1.3)
		5	0.753	0.691	0.597	0.516	6.0(2.8)	6.6(2.4)	6.1(2.9)	6.6(2.4)
		3	0.489	0.259	0.332	0.147	5.0(3.8)	6.2(3.6)	5.4(4.3)	6.2(3.6)
	0.20	20	0.481	0.546	0.359	0.418	20.0(8.5)	20.0(5.3)	20.0(9.0)	19.9(5.4)
		10	0.368	0.344	0.258	0.228	14.3(9.9)	15.1(7.2)	15.7(10.7)	15.2(7.2)
		5	0.218	0.141	0.138	0.084	13.2(12.0)	15.0(9.5)	15.9(12.7)	15.0(9.4)
	0.35	20	0.890	0.926	0.821	0.874	20.0(4.6)	20.0(2.8)	20.0(4.7)	20.0(2.9)
40		10	0.766	0.742	0.659	0.606	11.9(6.0)	13.2(4.9)	12.2(6.5)	13.2(4.9)
		5	0.468	0.259	0.336	0.150	9.6(8.9)	12.5(7.9)	10.9(10.1)	12.5(7.9)
	0.50	20	0.995	0.998	0.989	0.995	20.0(2.3)	20.0(1.6)	20.0(2.2)	20.0(1.6)
		10	0.972	0.966	0.943	0.921	10.9(3.1)	12.2(3.3)	10.9(3.1)	12.1(3.4)
		5	0.770	0.428	0.630	0.254	7.3(5.7)	10.8(6.6)	7.7(6.4)	10.9(6.6)
	0.20	35	0.693	0.778	0.581	0.671	35.0(12.2)	35.0(7.0)	35.0(12.7)	35.0(7.0)
		50	0.596	0.626	0.469	0.493	45.9(14.1)	44.2(9.6)	44.8(15.1)	44.1(9.5)
		60	0.377	0.268	0.262	0.160	50.1(18.5)	46.2(14.5)	46.8(20.6)	46.2(14.6)
	0.35	35	0.987	0.994	0.974	0.987	35.0(5.0)	35.0(3.2)	35.0(5.0)	35.0(3.2)
70		50	0.964	0.971	0.934	0.938	48.8(6.1)	46.9(5.4)	48.8(6.1)	46.9(5.4)
		60	0.795	0.616	0.684	0.440	56.3(10.4)	50.8(10.9)	55.7(11.6)	50.8(10.8)
	0.50	35	1.000	1.000	1.000	1.000	35.0(2.1)	35.0(1.7)	35.0(2.2)	35.0(1.7)
		50	1.000	1.000	0.999	1.000	49.5(2.6)	48.1(3.4)	49.5(2.5)	48.1(3.4)
		60	0.983	0.922	0.964	0.810	58.6(4.8)	53.4(8.4)	58.6(4.8)	53.5(8.3)
	0.20	50	0.827	0.896	0.743	0.831	50.0(14.4)	50.0(8.0)	49.9(14.6)	50.0(8.0)
		70	0.753	0.793	0.648	0.692	66.4(16.5)	63.8(11.2)	65.6(17.4)	63.7(11.1)
		90	0.366	0.206	0.258	0.127	75.0(27.5)	66.7(22.2)	69.6(31.0)	66.8(22.4)
	0.35	50	0.999	1.000	0.998	0.999	50.0(4.9)	50.0(3.4)	50.0(4.9)	50.0(3.4)
100		70	0.996	0.997	0.990	0.993	69.1(5.8)	66.9(5.8)	69.1(5.9)	67.0(5.8)
		90	0.795	0.471	0.689	0.303	85.0(14.9)	73.9(17.6)	84.1(16.8)	73.8(17.6)
	0.50	50	1.000	1.000	1.000	1.000	50.0(2.1)	50.0(1.7)	50.0(2.1)	50.0(1.7)
		70	1.000	1.000	1.000	1.000	69.6(2.3)	68.2(3.4)	69.6(2.3)	68.2(3.4)
		90	0.986	0.808	0.968	0.607	88.3(6.5)	77.9(14.2)	88.2(6.5)	77.9(14.1)
	0.20	75	0.943	0.974	0.905	0.951	74.9(15.6)	75.0(8.9)	75.0(16.0)	75.0(9.0)
		100	0.915	0.947	0.862	0.903	97.6(17.3)	94.3(12.0)	97.4(17.8)	94.3(12.2)
		125	0.729	0.637	0.624	0.490	116.6(26.6)	106.5(22.8)	114.4(29.8)	106.6(22.9)
	0.35	75	1.000	1.000	1.000	1.000	75.0(4.6)	75.0(3.6)	75.0(4.7)	75.0(3.6)
150		100	1.000	1.000	1.000	1.000	99.5(4.9)	97.5(5.4)	99.5(4.9)	97.5(5.4)
		125	0.995	0.987	0.989	0.964	123.4(7.7)	114.8(14.1)	123.4(8.0)	114.9(14.2)
	0.50	75	1.000	1.000	1.000	1.000	75.0(2.0)	75.0(1.8)	75.0(2.0)	75.0(1.8)
		100	1.000	1.000	1.000	1.000	99.7(2.1)	98.6(3.0)	99.7(2.1)	98.6(3.0)
		125	1.000	1.000	1.000	1.000	124.3(2.9)	118.4(9.7)	124.3(2.8)	118.4(9.7)

Table3. The Monte Carlo powers (at $\alpha = 0.05$) of the tests RD, RK, RDD, RKK, and the Monte Carlo means and standard deviations (Std) of the estimators $\hat{\nu}_D$, $\hat{\nu}_K$, $\hat{\nu}_{DD}$, $\hat{\nu}_{KK}$, when $F_1 = LogNorm(0,1)$, $F_2 = LogNorm(\beta,1)$, for different sample sizes n and values of β and ν. Observations with the subscript $\nu - 1$ are the last observations before the change.

n	β	$\nu-1$	Power of RD	Power of RK	Power of RDD	Power of RKK	Mean (Std) of $\hat{\nu}_D - 1$	Mean (Std) of $\hat{\nu}_K - 1$	Mean (Std) of $\hat{\nu}_{DD} - 1$	Mean (Std) of $\hat{\nu}_{KK} - 1$
	0.7	10	0.323	0.356	0.210	0.251	10.0(4.6)	9.9(3.3)	9.9(4.9)	9.9(3.3)
		5	0.241	0.217	0.148	0.131	7.8(5.2)	8.0(4.1)	8.5(5.5)	8.0(4.1)
		3	0.163	0.121	0.091	0.076	7.4(5.8)	7.8(4.8)	8.5(6.0)	7.8(4.8)
	1.2	10	0.662	0.720	0.526	0.597	10.0(3.0)	9.9(2.0)	10.0(3.2)	9.9(2.0)
20		5	0.522	0.461	0.360	0.312	6.6(3.8)	7.1(3.1)	7.0(4.1)	7.1(3.0)
		3	0.325	0.195	0.189	0.110	5.8(4.7)	6.8(4.1)	6.7(5.3)	6.8(4.1)
	1.7	10	0.907	0.936	0.831	0.887	10.0(1.8)	10.0(1.2)	10.0(1.8)	10.0(1.2)
		5	0.785	0.724	0.640	0.563	6.0(2.6)	6.5(2.3)	6.0(2.7)	6.5(2.3)
		3	0.546	0.273	0.353	0.154	4.8(3.6)	6.1(3.5)	5.2(4.1)	6.1(3.5)
	0.7	20	0.516	0.584	0.395	0.464	20.0(8.0)	19.9(5.0)	20.0(8.5)	19.9(5.0)
		10	0.399	0.377	0.287	0.256	14.0(9.5)	14.9(6.9)	15.1(10.3)	14.9(7.0)
		5	0.235	0.149	0.147	0.089	12.8(11.6)	14.7(9.3)	15.2(12.6)	14.6(9.3)
	1.2	20	0.915	0.946	0.859	0.904	20.0(4.1)	20.0(2.6)	20.0(4.2)	20.0(2.6)
40		10	0.808	0.782	0.711	0.663	11.6(5.4)	13.0(4.5)	11.8(5.8)	13.0(4.6)
		5	0.520	0.280	0.375	0.162	9.0(8.4)	12.1(7.7)	10.1(9.5)	12.1(7.7)
	1.7	20	0.997	0.999	0.993	0.996	20.0(2.0)	20.0(1.4)	20.0(2.0)	20.0(1.4)
		10	0.979	0.973	0.956	0.936	10.8(2.8)	12.0(3.2)	10.8(2.9)	12.0(3.2)
		5	0.814	0.453	0.687	0.269	7.1(5.4)	10.6(6.4)	7.3(5.9)	10.6(6.4)
	0.7	35	0.734	0.815	0.625	0.722	35.0(11.3)	35.0(6.4)	35.1(11.7)	35.0(6.5)
		50	0.640	0.669	0.517	0.539	46.3(13.0)	44.6(9.1)	45.6(14.3)	44.5(9.1)
		60	0.411	0.292	0.290	0.180	51.0(17.7)	46.8(14.2)	48.2(19.8)	46.8(14.2)
	1.2	35	0.992	0.997	0.984	0.993	35.0(4.3)	35.0(2.9)	35.0(4.4)	35.0(2.9)
70		50	0.976	0.981	0.953	0.960	49.0(5.2)	47.2(5.1)	49.0(5.3)	47.1(5.0)
		60	0.841	0.672	0.747	0.495	57.0(9.4)	51.4(10.3)	56.6(10.2)	51.4(10.4)
	1.7	35	1.000	1.000	1.000	1.000	35.0(2.0)	35.0(1.6)	35.0(1.9)	35.0(1.6)
		50	1.000	1.000	1.000	1.000	49.5(2.3)	48.2(3.2)	49.6(2.3)	48.2(3.2)
		60	0.987	0.938	0.972	0.849	58.8(4.2)	53.7(8.1)	58.8(4.3)	53.8(8.0)
	0.7	50	0.866	0.924	0.795	0.871	50.0(12.8)	49.9(7.3)	50.0(13.2)	50.0(7.3)
		70	0.796	0.835	0.705	0.743	66.9(15.1)	64.3(10.6)	66.4(15.8)	64.2(10.5)
		90	0.403	0.224	0.291	0.136	76.7(26.2)	67.6(21.8)	72.0(29.7)	67.4(21.9)
	1.2	50	1.000	1.000	0.999	1.000	50.0(4.2)	50.0(3.1)	50.0(4.3)	50.0(3.0)
100		70	0.998	0.999	0.994	0.996	69.3(5.0)	67.3(5.3)	69.4(5.0)	67.2(5.2)
		90	0.843	0.520	0.751	0.343	85.8(13.1)	74.6(17.0)	85.2(14.8)	74.6(17.0)
	1.7	50	1.000	1.000	1.000	1.000	50.0(1.9)	50.0(1.6)	50.0(1.9)	50.0(1.6)
		70	1.000	1.000	1.000	1.000	69.6(2.1)	68.3(3.1)	69.6(2.1)	68.4(3.2)
		90	0.989	0.843	0.975	0.660	88.6(5.6)	78.4(13.9)	88.5(5.7)	78.3(13.9)
	0.7	75	0.962	0.985	0.935	0.969	75.0(13.8)	75.0(8.1)	75.0(14.0)	75.0(8.2)
		100	0.940	0.964	0.899	0.931	98.1(15.3)	94.9(11.0)	98.0(15.9)	94.9(11.0)
		125	0.776	0.686	0.684	0.541	117.9(23.8)	107.8(22.0)	116.4(26.7)	107.6(22.0)
	1.2	75	1.000	1.000	1.000	1.000	75.0(4.1)	75.0(3.2)	75.0(4.0)	75.0(3.2)
150		100	1.000	1.000	1.000	1.000	99.6(4.2)	97.7(4.9)	99.6(4.3)	97.8(4.9)
		125	0.997	0.992	0.994	0.978	123.7(6.6)	115.6(13.3)	123.7(6.5)	115.5(13.3)
	1.7	75	1.000	1.000	1.000	1.000	75.0(1.9)	75.0(1.6)	75.0(1.8)	75.0(1.6)
		100	1.000	1.000	1.000	1.000	99.7(1.9)	98.7(2.8)	99.7(2.0)	98.7(2.8)
		125	1.000	1.000	1.000	1.000	124.4(2.5)	118.9(9.3)	124.4(2.5)	118.8(9.3)

Tables 1-3 show that for all considered examples and sample sizes, when an alternative is one-sided, the test RK is preferable to the test RD from the power perspective if a change in the distributions occurs in the middle ($\nu \approx n/2$) and the situation is reversed when the change occurs in the edges ($\nu \approx 2$ or $\nu \approx n$). Thus, our simulation results confirm the conclusions follow from the Monte Carlo experiments presented by Wolfe and Schechtman (1984) and Gurevich (2009) about power comparisons of the nonparametric tests RK and RD. Note also that when an alternative is two-sided, the power's comparison of the tests RKK and RDD seems to be a similar to that of RK and RD. That is the test RKK is preferable to the test RDD if the real change occurs in the middle and the test RDD is preferable to the test RKK when the real change point ν is close to the edges. As expected, the powers of the test RK, RD are essentially higher than that of the tests RKK, RDD, respectively, for all considered situations. Obviously, it follows from the fact that the tests RK and RD have been constructed utilizing an additional information about possible alternatives. Finally, analyzing the simulated powers of the tests RD, RK, RDD, RKK presented in Tables 1-3, we conclude that all these test are very efficient providing rather high powers even for average and small sample sizes ($n \approx 40$, $n \approx 20$) and insignificant real changes in distributions of the observations.

It ***follows from*** Tables 1-3 that the simulated means and standard deviations of the estimators $\hat{\nu}_K$ are very similar to that of the estimator $\hat{\nu}_{KK}$ for all considered examples and values of the parameter ν. That means that if one decides to use the estimator of a change point ν based on the statistic K, it is not important to define the alternative in (1) as a one-sided (and use the estimator $\hat{\nu}_K$) or as a two-sided (and use the estimator $\hat{\nu}_{KK}$). Comparing the behavior of the estimator $\hat{\nu}_D$ with that of the estimator $\hat{\nu}_{DD}$ shows that the estimator $\hat{\nu}_D$ is more exact than $\hat{\nu}_{DD}$ (i.e., the simulated means of $\hat{\nu}_D$ are closer to the real values of ν and his simulated standard deviations are less than that of $\hat{\nu}_{DD}$), especially for large sample sizes. That is, if one decides to use the statistic D for the testing of the hypotheses (1) and estimating of the change point ν (i.e., the possible change is suspected to be close to edges), the definition of the alternative of (1) as a one-sided not only increases the power of the appropriate test (RD is more powerful than RDD) but also yields more exact estimator ($\hat{\nu}_D$ is more exact than $\hat{\nu}_{DD}$).

In addition, Tables 1-3 demonstrate that for all considered examples and values of ν, for small and average sample sizes ($n = 20,40$) the estimator $\hat{\nu}_D$ is slightly less biased than the estimator $\hat{\nu}_K$ (i.e., the simulated means of the estimator $\hat{\nu}_D$ are slightly closer to the real values of ν than that of the estimator $\hat{\nu}_K$) but has a slightly higher variance than that of $\hat{\nu}_K$. For large sample sizes, the estimator $\hat{\nu}_D$ is essentially less biased than the estimator $\hat{\nu}_K$ (especially if a real changes are close to the edges), and has a higher variance than the estimator $\hat{\nu}_K$ for $\nu \approx n/2$, and a lower variance than $\hat{\nu}_K$ for $\nu \approx 2$ and $\nu \approx n$. Thus, the estimator $\hat{\nu}_D$ seems to be preferable to all the other estimators considered here and can be recommended to be applied in practice for estimation of a change point parameter, even for the situations where the possible change is suspected to be in the middle and tests based on the statistic K are appropriate for the hypotheses (1). Moreover, the simulated results presented in Tables 1-3 demonstrate a good performance of all four proposed estimators from the practical point of view especially for average and large sample sizes.

As aforementioned, the asymptotic analysis of the estimators' behavior is beyond the scope of this article, however, it seems from Tables 1-3 that the estimators $\hat{\nu}_K$, $\hat{\nu}_D$, $\hat{\nu}_{KK}$, $\hat{\nu}_{DD}$ are consistent when $\nu \to \infty$, $n - \nu \to \infty$.

4. Conclusions

In this article we have reviewed the recent change point literature related to the retrospective change point detection and estimation issues. We focused on the problem of nonparametric estimation of the change point parameter and considered four relevant estimators. We conducted a broad Monte Carlo study for judging the accuracy of the proposed change point estimators also comparing them from an implementation point of view. Simulation results confirm the efficiency of these estimators even for small and average sample sizes. Specific practical recommendations for using different estimators in varied situations are given in Section 3. Thus, we believe that the outputs of this manuscript have great potential to be applied in practice.

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