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ANALYTICAL AND NUMERICAL STUDIES OF PERTURBED RENEWAL EQUATIONS WITH MULTIVARIATE NON-POLYNOMIAL PERTURBATIONS

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Abstract: *The object of study is a model of nonlinearly perturbed continuous-time renewal equation with multivariate non-polynomial perturbations. The characteristics of the distribution generating the renewal equation are assumed to have expansions in a perturbation parameter with respect to a non-polynomial asymptotic. Exponential asymptotics for such a model as well as their applications are given. Numerical studies are performed to gain insights into the asymptotical results.*

Key words: *perturbed renewal equation; nonlinear perturbation; non-polynomial perturbation; perturbed risk process; ruin probability*

1. Introduction

This paper deals with nonlinearly perturbed renewal equations with a new type of non-polynomial perturbations. That is, some characteristics of the distribution generating the perturbed renewal equation, namely the defect and moments, can be expanded in the perturbation parameter ε up to some order α with respect to the following non-standard non-polynomial asymptotic scale,

$$\{\varphi_{\vec{n}}(\varepsilon) = \varepsilon^{\vec{n}\cdot\vec{\omega}}, \vec{n} \in \mathbf{N}_0^k\}, \text{ as } \varepsilon \to 0, \tag{1}$$

where $\mathbf{N}_0$ is the set of non-negative integers, $\mathbf{N}_0^k \equiv \mathbf{N}_0 \times \cdots \times \mathbf{N}_0, 1 \le k < \infty$ with the product being taken k times, and $\vec{\omega}$ is a parameter vector of dimension k. In (1), $\vec{n}\cdot\vec{\omega}$ denotes the dot product of vector $\vec{n}$ and $\vec{\omega}$, and by the definition of asymptotic scale, the gauge functions $\varphi_{\vec{n}}(\varepsilon)$ are ordered by index $\vec{n}$ in such way that the later function in the sequence is always o-function of the previous one. Further, we assume that the parameter vector $\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_k)$ has the following properties: **(i)** $1 = \omega_1 < \omega_2 < \dots < \omega_k$; **(ii)** the components are linearly independent over the field Q of rational numbers, i.e., ω_i / ω_j is an irrational number for any $i \neq j, i, j = 1, \dots, k$. Note that it follows from **(i)** and **(ii)** that $\omega_2, \dots, \omega_k$ are irrational numbers. Throughout the paper, the symbol $\vec{\omega}$ refers to some parameter vector satisfying these two properties.

The aim of this paper is to present the asymptotic behavior of such perturbed renewal equations, illustrate the result by applications and carry out numerical studies of the applications. The case for $k = 2$ in (1) has been studied in the previous research (Ni, Silvestrov and Malyarenko 2008). Setting $k = 1$, the asymptotic scale (1) reduces to the standard polynomial asymptotic scale, and this case was first investigated in Silvestrov (1995). The present paper covers the general case where k can be any finite positive integer, that is, the case with "multivariate" non-polynomial perturbations. Other works on nonlinearly perturbed renewal equation with non-polynomial perturbations have been done by Englund and Silvestrov (1997) and Englund (2001), where the expansions of defect and moments have polynomial and mixed polynomial-exponential forms.

For a general theory of nonlinearly perturbed renewal equations with applications to non-linearly perturbed stochastic systems, we refer to the book by Gyllenberg and Silvestrov (2008) and references therein. Note that all expansions in this book are based on the standard polynomial asymptotical scale.

2. The Model

Let us consider the following perturbed renewal equation which holds for every $\varepsilon \geq 0$:

$$x_\varepsilon(t) = q_\varepsilon(t) + \int_0^t x_\varepsilon(t-s) F_\varepsilon(ds), \;\; t \geq 0, \tag{2}$$

where the force function $q_\varepsilon(t)$ refers to a measurable real-valued function on $[0, \infty)$ being bounded on every finite interval. The distribution function $F_\varepsilon(\cdot)$ generating this renewal equation has its support on $[0, \infty)$, is not concentrated at 0 and can be improper. It is known that there exists a unique solution which is both measurable and bounded on every finite interval solution, $x_\varepsilon(t)$, for equation (2).

The defect and moments for F_ε are defined as

$$f_\varepsilon = 1 - F_\varepsilon(\infty), \qquad m_{\varepsilon r} = \int_0^\infty s^r F_\varepsilon(ds), \qquad r \geq 1.$$

Assume that the following perturbation conditions hold for $F_\varepsilon(\cdot)$ and $q_\varepsilon(\cdot)$.

A. $F_\varepsilon(t) \Rightarrow F_0(t)$ as $\varepsilon \to 0$, where $F_0(t)$ is a proper and non-arithmetic distribution function.

B. (*Cramér type condition*) There exists $\delta > 0$ such that

$\overline{\lim}_{0\le\varepsilon\to 0}\int_0^\infty e^{\delta s}F_\varepsilon(ds)<\infty$.

C. (a) $\lim_{u\to 0}\overline{\lim}_{0\le\varepsilon\to 0}\sup_{|v|\le u}|q_\varepsilon(t+v)-q_0(t)|=0$ a.e. with respect to Lebesgue measure on $[0,\infty)$;

(b) $\overline{\lim}_{0\le\varepsilon\to 0}\sup_{0\le t\le T}|q_\varepsilon(t)|<\infty$ for every $T\ge 0$;

(c) $\lim_{T\to\infty}\overline{\lim}_{0\le\varepsilon\to 0}h\sum_{r\ge T/h}\sup_{rh\le t\le(r+1)h}e^{\gamma t}|q_\varepsilon(t)|=0$ for some $h>0$, $\gamma>0$.

Note that symbol $F_\varepsilon(\cdot)\Rightarrow F_0(\cdot)$ as $\varepsilon\to 0$ denotes weak convergence of the distribution functions. Notations $\overline{\lim}$ and $\underline{\lim}$ are equivalent to $\limsup$ and $\liminf$, respectively.

It follows from condition B that the exponential moment of F_ε, defined as

$$\phi_\varepsilon(\rho)=\int_0^\infty e^{\rho s}F_\varepsilon(ds),\ \rho\ge 0,$$

is finite for $\rho<\delta$ and ε small enough.

It is known that, under condition A and B there is a unique nonnegative root, ρ_ε, of the following characteristic equation:

$$\phi_\varepsilon(\rho)=\int_0^\infty e^{\rho s}F_\varepsilon(ds)=1, \tag{3}$$

for ε small enough and $\rho_\varepsilon\to 0$ as $\varepsilon\to 0$.

The following theorem (Silvestrov, 1976, 1978, 1979) serves as the starting point for the present study.

THEOREM 1

Let conditions A, B and C be satisfied. Then for any $0\le t_\varepsilon\to\infty$ as $\varepsilon\to 0$, the following asymptotical relation holds

$$\frac{x_\varepsilon(t_\varepsilon)}{\exp\{-\rho_\varepsilon t_\varepsilon\}}\to x_0(\infty)=\frac{\int_0^\infty q_0(s)ds}{m_{01}}\quad\text{as }\ \varepsilon\to 0. \tag{4}$$

By condition A we have $f_\varepsilon\to f_0=0$ as $\varepsilon\to 0$ and by Condition B all moments of F_ε are finite, i.e. $m_{\varepsilon r}<\infty, r\ge 1$. Condition A and B also imply that for ε small enough, $m_{\varepsilon r}\to m_{0r}\in(0,\infty)$ as $\varepsilon\to 0, r\ge 1$. The basic idea of the present research is: by assuming some appropriate form of asymptotic expansions for f_ε and $m_{\varepsilon r}$, the corresponding asymptotic expansion of ρ_ε may be obtained, which can be used to improve the asymptotic relation (4) to a more explicit form.

For some real number $\alpha \geq 1$, notation $[\alpha]_{\vec{\omega}}$ is defined as: $[\alpha]_{\vec{\omega}} = \max(\vec{n} \cdot \vec{\omega} : \vec{n} \cdot \vec{\omega} \leq \alpha, \vec{n} \in \mathbf{N}_0^k)$, i.e. the last gauge function in (1) that has the order less than or equal to α is $\varepsilon^{[\alpha]_{\vec{\omega}}}$.

Given α and a specific parameter vector $\vec{\omega}$, by property (ii) of $\vec{\omega}$ we know that there exists a unique vector $\vec{n}$ such that $[\alpha]_{\vec{\omega}} = \vec{n} \cdot \vec{\omega}$, we denote this $\vec{n}$ by $\vec{f}(\alpha, \vec{\omega})$.

Notation $[\alpha]$ is used to denote the integer part of number α. Let us also define the following two sets:

$$\mathbf{R}_i(\vec{n}) = \{\vec{p} : p_1 \leq n_1, \ldots, p_k \leq n_k, \sum_{j=1}^{k} p_j \geq i\}, \quad \mathbf{R}_i'(\vec{n}) = \mathbf{R}_i(\vec{n}) \setminus \{\vec{n}\},$$

where $\vec{n}, \vec{p} \in \mathbf{N}_0^k$. For example, if $\vec{n} = (2,1)$, $i = 2$ then $\mathbf{R}_i(\vec{n})$ refers to the set $\{(1,1),(2,0),(2,1)\}$ while $\mathbf{R}_i'(\vec{n})$ represents the set $\{(1,1),(2,0)\}$.

All vectors in this paper are k-dimensional (as for $\vec{\omega}$) row vectors unless stated otherwise, and they are represented with lowercase Roman/Greek letters with right-pointing arrows above. Symbol $\vec{0}$ is a vector with all components equal to zeros, and $\vec{e}_i$ refers to i-th unit vector, i.e. all components are zero except that the i-th component is equal to one.

We are now in a position to impose the following additional perturbation conditions which hold for a given real number $\alpha \geq 1$ and for some given parameter vector $\vec{\omega}$.

$\mathbf{P}_{\vec{\omega}}^{(\alpha)}$: (a) $1 - f_\varepsilon = 1 + \sum_{1 \leq \vec{n} \cdot \vec{\omega} \leq \alpha} b_{\vec{n},0} \varepsilon^{\vec{n} \cdot \vec{\omega}} + o(\varepsilon^{[\alpha]_{\vec{\omega}}})$, where all coefficients are finite.

(b) $m_{\varepsilon r} = m_{0r} + \sum_{1 \leq \vec{n} \cdot \vec{\omega} \leq \alpha - r} b_{\vec{n},r} \varepsilon^{\vec{n} \cdot \vec{\omega}} + o(\varepsilon^{[\alpha - r]_{\vec{\omega}}})$, for $r = 1, \ldots, [\alpha]$, where all coefficients are finite.

Remark 1. In condition $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$, the defect and moments are expanded, up to order α, with respect to asymptotic scale (1). For convenience, notation: $b_{\vec{0},0} = 1$, $b_{\vec{0},r} = m_{0r}$ is also used.

3. The Main Result

The following theorem presents the exponential asymptotics for the solution to the perturbed renewal equation described in the previous section.

THEOREM 2

Let conditions A, B and $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ *be satisfied. Then:*

(i) *There exists a unique non-negative solution to characteristic equation (3) for all* ε *that are small enough. Further, the following expansion for* ρ_ε *holds,*

$$\rho_\varepsilon = \sum_{1\le \vec{n}\cdot\vec{\omega}\le\alpha} a_{\vec{n}}\varepsilon^{\vec{n}\cdot\vec{\omega}} + o(\varepsilon^{[\alpha]_{\vec{\omega}}}), \tag{5}$$

where the coefficients can be calculated by the following recurrent formula: $a_{\vec{e}_1} = -b_{\vec{e}_1,0}\big/ b_{\vec{0},1}$, *and in general for* $\vec{n}: 1 < \vec{n}\cdot\vec{\omega} \le \alpha$,

$$a_{\vec{n}} = -\frac{1}{b_{\vec{0},1}}(b_{\vec{n},0} + \sum_{\vec{p}\in\mathbf{R}'_1(\vec{n})} b_{\vec{n}-\vec{p},1}a_{\vec{p}} + \sum_{i=2}^{n_1+n_2+\ldots+n_k} \sum_{\vec{p}\in\mathbf{R}_i(\vec{n})} b_{\vec{n}-\vec{p},i}(\sum_{\vec{j}(\vec{p})\in\mathbf{D}_i(\vec{p})} \prod_{\vec{r}\in\mathbf{R}'_1(\vec{p})} \frac{(a_{\vec{r}})^{j_{\vec{r}}}}{(j_{\vec{r}})!})), \tag{6}$$

where $\mathbf{D}_i(\vec{p})$ *is the set of all nonnegative and integer solutions,* $\vec{j}(\vec{p}) \equiv (j_{\vec{r}}, \vec{r}\in\mathbf{R}'_1(\vec{p}))$, *for the Diophantine system*

$$\begin{cases} \sum_{\vec{r}\in\mathbf{R}'_1(\vec{p})} j_{\vec{r}} = i, \\ \sum_{\vec{r}\in\mathbf{R}'_1(\vec{p})} \vec{r}\times j_{\vec{r}} = \vec{p} \end{cases} \tag{7}$$

(ii) *If the coefficients for the defect satisfy* $b_{\vec{n},0} = 0$ *for* $\vec{n}$ *such that* $\vec{n}\cdot\vec{\omega}\le\beta$ *for some* $1\le\beta\le\alpha$, *then* $a_{\vec{n}} = 0$ *for* $\vec{n}$ *such that* $\vec{n}\cdot\vec{\omega}\le\beta$.

(iii) *If in addition condition C holds, then for any* $0\le t_\varepsilon\to\infty$ *balanced with* $\varepsilon\to 0$ *in such a way that* $\varepsilon^{[\beta]_{\vec{\omega}}} t_\varepsilon \to \lambda_\beta \in [0,\infty)$ *where* $\beta\in[1,\alpha]$ *is a given real number, we have the following asymptotical relation:*

$$\exp\{(\sum_{1\le\vec{n}\cdot\vec{\omega}<[\beta]_{\vec{\omega}}} a_{\vec{n}}\,\varepsilon^{\vec{n}\cdot\vec{\omega}})t_\varepsilon\}\, x_\varepsilon(t_\varepsilon) \to e^{-\lambda_\beta a^{(1)}} x_0(\infty) \quad \text{as} \quad \varepsilon\to 0,$$

where $a^{(1)} = a_{\vec{p}}$ *with* $\vec{p} = \vec{f}(\beta,\vec{\omega})$.

Remark 2. The coefficient $a_{\vec{n}}$ can be calculated from the recurrent formula (6) if $\vec{n}$ satisfies $1\le\vec{n}\cdot\vec{\omega}\le\alpha$. It can be directly seen from formula (6) that $a_{\vec{n}}$ depends on the set of coefficients $\{a_{\vec{p}}:\ \vec{p}\in\mathbf{R}'_1(\vec{n})\}$ which is obviously a subset to $\{a_{\vec{p}}: 1\le\vec{p}\cdot\vec{\omega}<\vec{n}\}$. Also one can observe from (5) and (6) that the value of coefficient $a_{\vec{n}}$ does not depend on parameter vector $\vec{\omega}$ and parameter α .

Remark 3. For a given $\vec{\omega}$ and a given α , the expansion of ρ_ε (5) takes a unique form, and so is the sequence of coefficients $a_{\vec{n}}$, $1\le\vec{n}\cdot\vec{\omega}<\alpha$. Let the terms in expansion (5) be ordered in terms of the powers of ε, a natural choice of recursive algorithm would be to first calculate the first-by-order coefficient in the expansion then the second-by-order coefficient and so on.

Remark 4. If the dimension of $\vec{\omega}$ is one , so that $\vec{\omega} = 1$, and let also α be some positive integer greater than one, we have the particular case where the defect and moments are expanded with respect to the standard polynomial asymptotic scale, up to and including the order α . This case has been studied in Silvestrov (1995) and Gyllenberg and Silvestrov (2008). Theorem 2 reduces, in this case, to the corresponding result obtained there. Similarly when $\vec{\omega}$ has dimension 2, so that $\vec{\omega}=(1,\omega)$ for some irrational $\omega>1$, and let α be some real number, Theorem 2 reduces, in this case, to the corresponding result in Ni, Silvestrov and Malyarenko (2008). For convenience, we shall call the latter case, i.e. the case when $\vec{\omega}$ has dimension 2, as the "bivariate" case.

One can use recurrent formula (6) to calculate manually the coefficients $a_{\vec{n}}$ when parameter k and α are relatively small. For larger values of k and α, it is better to program formula (6). For instance, the corresponding MATLAB routine has been developed by the author. The algorithm takes inputs α and $\vec{\omega}$ (hence the dimension of $\vec{\omega}$, k), then determine the sequence of coefficients included in (5) by solving $\vec{n}\cdot\vec{\omega} \le \alpha$ for integer values of $\vec{n}$ and sorting the solutions, $\vec{n}$, in ascending order with respect to the value of $\vec{n}\cdot\vec{\omega}$. The next step is to determine recursively the coefficients using formula (6). Although tedious, most part of this formula are relatively easy to program. Let us only describe briefly the algorithm for solving the Diophantine system (7).

The second equation in system (7) leads to k equations since the dimension of vector $\vec{r}$ and $\vec{p}$ is k. The unknowns are $j_{\vec{r}}, \vec{r}\in\mathbf{R}_1'(\vec{p})$. Denote $q=|\mathbf{R}_1'(\vec{p})|$, i.e. q is the number of vectors in the set $\mathbf{R}_1'(\vec{p})$. System (7) is indeed a Diophantine system of q unknowns in $k+1$ equations. Let us express this system in the matrix equation $\mathbf{A}\vec{x}=\vec{b}$, where $\mathbf{A}$ is $(k+1)\times q$ matrix, i.e. the matrix of coefficients for the system, $\vec{x}$ is the unknown column vector with q entries, and b is a column vector with $k+1$ entries. The problem is therefore: determine the set of non-negative integer solutions to $\mathbf{A}\vec{x}=\vec{b}$, and this can be efficiently solved by using a recursive algorithm.

4. Applications

The results may have many potential applications, for instance, to the analysis of nonlinearly perturbed risk processes and processes which are used to describe functioning of queueing systems. We present in this section two examples of perturbed classical risk processes, with bivariate and multivariate non-polynomial perturbations respectively. Theorem 2 is applied to obtain asymptotic behaviour for ruin probabilities and experimental numerical studies are carried out to gain insights into the asymptotical results. Since there's a duality of classical risk processes with the workload process of a $M/G/1$ queue, and with the dam/storage process, the results also have interpretation in these areas.

Let us consider the perturbed classical risk-process which describes the time evolution of the reserves of an insurance company

$$X_\varepsilon(t)=ct-\sum_{j=1}^{N(t)}Z_{j\varepsilon}, \qquad t\ge 0, \tag{8}$$

where $c>0$ is the gross risk premium rate; $N(t), t\ge 0$ is the Poisson claim arrival process with rate λ; the claim sizes $Z_{j\varepsilon},\ j=1,\dots,N(t)$ are i.i.d. nonnegative random variables, independent of process $N(t)$, that follow a common distribution $G_\varepsilon(z)$ with a finite mean $\mu_\varepsilon=\int_0^\infty z\,G_\varepsilon(dz)<\infty$;

It is usually assumed that the moment characteristics of $G_\varepsilon(z)$ depend on the perturbation parameter ε but converge in some sense to the corresponding characteristics of limiting distribution $G_0(z)$ as $\varepsilon \to 0$. These continuity conditions allow us to consider the risk process $X_\varepsilon(t)$ for $\varepsilon > 0$ as a perturbed version of $X_0(t)$ for $\varepsilon = 0$.

The loading rate of claims are characterized by a constant α_ε or equivalently by the safety loading coefficient η_ε, defined respectively as

$$\alpha_\varepsilon = \frac{\lambda \mu_\varepsilon}{c}; \qquad \eta_\varepsilon = \frac{1-\alpha_\varepsilon}{\alpha_\varepsilon}. \tag{9}$$

Let $u \geq 0$ be the initial reserve of the insurance company, the object of our study is the ruin probability,

$$\Psi_\varepsilon(u) = \mathrm{P}\{u + \inf_{t \geq 0} X_\varepsilon(t) < 0\}.$$

The ruin probability is known to be equal to one if $\alpha_\varepsilon \geq 1$ or equivalently if the safety loading $\eta_\varepsilon \leq 0$. For $\alpha_\varepsilon \leq 1$, $\Psi_\varepsilon(u)$ as a function of initial reserve u, satisfies the following perturbed renewal equation (Feller, 1966),

$$\Psi_\varepsilon(u) = \alpha_\varepsilon(1 - \widetilde{G}_\varepsilon(u)) + \alpha_\varepsilon \int_0^u \Psi_\varepsilon(u-s)\widetilde{G}_\varepsilon(ds), \qquad u \geq 0, \tag{10}$$

where $\widetilde{G}_\varepsilon(u)$ is the integrated tail distribution, i.e.

$$\widetilde{G}_\varepsilon(u) = \frac{1}{\mu_\varepsilon}\int_0^u (1 - G_\varepsilon(s))ds. \tag{11}$$

Note that $\alpha_\varepsilon = 1$ is the trivial case since $\Psi_\varepsilon(u) \equiv 1$ is a solution to equation (10) if $\alpha_\varepsilon = 1$.

The distribution function that generates the perturbed renewal equation (10) is $F_\varepsilon(u) = \alpha_\varepsilon \widetilde{G}_\varepsilon(u)$.

Denote α_0 as α_ε for $\varepsilon = 0$. Let us assume the following condition holds.

D. $\alpha_0 = 1$.

Note that condition **D** implies that $\Psi_0(u) = 1$ for all $u \geq 0$.

Our aim is to obtain the asymptotic behavior of $\Psi_\varepsilon(u)$ as the perturbation parameter $\varepsilon \to 0$ simultaneously as the initial reserve $u \to \infty$ under some balancing condition. Let us use notation u_ε to emphasize that u is changing together with ε.

4.1. Perturbed Risk Process with Bivariate Non-polynomial Perturbations

We consider the perturbed risk process (8) and assume the following form for the limiting claim size distribution $G_0(z)$,

$$G_0(z) = \begin{cases} 1 - \dfrac{(T_0 - z)^\omega}{T_0^\omega}, & 0 \leq z < T_0, \\ 1, & z \geq T_0, \end{cases} \tag{12}$$

where T_0 is a constant parameter and parameter $\omega > 1$ is some irrational number.

The first moment μ_0 for $G_0(z)$ is,

$$\mu_0 = \int_0^\infty s\, G_0(ds) = \frac{T_0}{\omega+1}. \tag{13}$$

Let the perturbed claim size distribution $G_\varepsilon(z)$ for $\varepsilon \ge 0$ be given by

$$G_\varepsilon(z) = \begin{cases} 1 - \dfrac{(T-z)^\omega}{T^\omega}, & 0 \le z < T, \\ 1, & z \ge T, \end{cases} \tag{14}$$

where T is a constant parameter and $T \le T_0$. Let us use $\varepsilon \equiv T_0 - T \ge 0$ as the perturbation parameter.

In other words, $G_\varepsilon(z) = \mathrm{P}\{Z_{j0} \wedge T \le z\}$ where Z_{j0}follows distribution $G_0(z)$, which can be caused for example by a excess-of-loss reinsurance with retention level T.

Taking account of (13), the first moment of $G_\varepsilon(z)$, μ_ε can be calculated as

$$\mu_\varepsilon = \int_0^\infty s G_\varepsilon(ds) = \mu_0 - \frac{\mu_0}{T_0^{\omega+1}} \varepsilon^{\omega+1}. \tag{15}$$

Note that it follows from (12) and (14) that $G_\varepsilon(z) \to G_0(z)$ as $\varepsilon \to 0$ for every $z \ge 0$. Also $\mu_\varepsilon \le \mu_0$ and $\mu_\varepsilon \to \mu_0$ as $\varepsilon \to 0$ due to (15).

It follows from condition **D**, (9), (13) and (15) that $\alpha_\varepsilon \le \alpha_0 = 1$ and $\alpha_\varepsilon \to \alpha_0 = 1$ as $\varepsilon \to 0$, which is the situation considered in a diffusion approximation for ruin probabilities.

Under condition **D** we have $\Psi_0(u) = 1$. Also we have $\alpha_\varepsilon \le 1$, hence the ruin probability $\Psi_\varepsilon(u)$ satisfies the perturbed renewal equation (10).

Since for $\varepsilon > 0$ we have $\alpha_\varepsilon < 1$, the distribution function $F_\varepsilon(u) = \alpha_\varepsilon \widetilde{G}_\varepsilon(u)$ in this case is improper, i.e. defect $f_\varepsilon = 1 - F_\varepsilon(\infty) = 1 - \alpha_\varepsilon > 0$ for $\varepsilon > 0$. Obviously $f_\varepsilon \to f_0 = 0$ as $\varepsilon \to 0$.

It can be shown that the defect f_ε takes the following form,

$$f_\varepsilon = 1 - \alpha_\varepsilon = 1 - \frac{\lambda \mu_\varepsilon}{c} = \frac{1}{T_0^{1+\omega}} \varepsilon^{1+\omega}. \tag{16}$$

By repeatedly applying integration by parts, the r-th moment, $r \ge 1$, of $F_\varepsilon(u)$ can be calculated as:

$$m_{\varepsilon r} = \int_0^\infty s^r F_\varepsilon(ds) = \frac{(r!) \cdot T_0^r}{\prod_{i=2}^{r+1} (\omega + i)} + \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \frac{\omega+1}{\omega+k+1} T_0^{r-k-\omega-1} \varepsilon^{k+1+\omega}. \tag{17}$$

Let us define $[\xi]_\omega \equiv \max(n + m\omega : n + m\omega \le \xi, n, m \in \mathbf{N}_0)$, where $\mathbf{N}_0$ is the set of non-negative integers. We now set $\xi = 4 + 3\omega$ so that $[\xi]_\omega = 4 + 3\omega$. Using (16), (17), the characteristics of $F_\varepsilon(u)$, namely the defect and moments, can be written down as the following perturbation condition.

$\mathbf{P}_{\omega}^{(\xi)}$: **(a)** $1-f_{\varepsilon}=1+\sum_{1\le n+m\omega\le 4+3\omega} b_{n,m,0}\varepsilon^{n+m\omega}+o(\varepsilon^{4+3\omega})$, where coefficients are given in (16);

(b) $m_{\varepsilon r}=m_{0r}+\sum_{1\le n+m\omega\le 4+3\omega-r} b_{n,m,r}\varepsilon^{n+m\omega}+o(\varepsilon^{4+3\omega-r})$, for $r=1,\dots,[4+3\omega]$, where coefficients are given in (17).

We would like to note that (16), (17) can be rewritten in the form of $\mathbf{P}_{\omega}^{(\xi)}$ for any $\xi<\infty$. Note also that the perturbation condition $\mathbf{P}_{\omega}^{(\xi)}$ is a particular case of condition $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ for the case $\vec{\omega}=(1,\omega)$ and $\alpha=\xi=4+3\omega$. It can also be shown that condition A, B, C hold for the perturbed renewal equation (10) with $G_{\varepsilon}(z)$ given by (14). Applying Theorem 2 we obtain the following exponential asymptotic expansion for the ruin probability.

THEOREM 3

*Let the claim distributions $G_0(z)$ and $G_{\varepsilon}(z)$ be given by formulas (12) and (14). Let also condition **D** holds and $\varepsilon=T_0-T\ge 0$ be the perturbation parameter. Then there exists a unique non-negative solution, ρ_{ε}, to the characteristic equation (3) and the following asymptotical relation holds,*

$$\rho_{\varepsilon}=a_{11}\,\varepsilon^{1+\omega}+a_{2,2}\,\varepsilon^{2+2\omega}+a_{3,2}\,\varepsilon^{3+2\omega}++a_{3,3}\,\varepsilon^{3+3\omega}+a_{4,3}\,\varepsilon^{4+3\omega}+o(\varepsilon^{4+3\omega}), \tag{18}$$

where

$$a_{1,1}=\frac{\omega+2}{T_0^{\omega+2}},\quad a_{2,2}=\frac{(\omega+2)^3}{(\omega+3)T_0^{2\omega+3}},\quad a_{3,2}=-\frac{(\omega+1)(\omega+2)}{T_0^{2\omega+4}},$$
$$a_{3,3}=\frac{(\omega+2)^4}{T_0^{3\omega+4}}\left(\frac{\omega+1}{(\omega+3)^2}+\frac{\omega+5}{2(\omega+3)(\omega+4)}\right),$$
$$a_{4,3}=-\frac{3(\omega+1)(\omega+2)^3}{(\omega+3)T_0^{3\omega+5}}.$$

(i) *For any $0\le u_{\varepsilon}\to\infty$ in such a way that $\varepsilon^{[\beta]_{\omega}}u_{\varepsilon}\to\lambda_{\beta}\in[0,\infty)$ for some $1+\omega\le\beta<2+2\omega$, the following asymptotical relation holds,*

$$\Psi_{\varepsilon}(u_{\varepsilon})\to\exp\{-\lambda_{\beta}a_{1,1}\}\quad as\ \varepsilon\to 0.$$

(ii) *For any $0\le u_{\varepsilon}\to\infty$ in such a way that $\varepsilon^{[\beta]_{\omega}}u_{\varepsilon}\to\lambda_{\beta}\in[0,\infty)$ for some $2+2\omega\le\beta<3+2\omega$, the following asymptotical relation holds,*

$$\exp\left\{\left(a_{1,1}\,\varepsilon^{1+\omega}\right)u_{\varepsilon}\right\}\Psi_{\varepsilon}(u_{\varepsilon})$$
$$\to\exp\left\{-\lambda_{\beta}\,a_{2,2}\right\}\quad as\ \varepsilon\to 0.$$

(iii) *For any $0\le u_{\varepsilon}\to\infty$ in such a way that $\varepsilon^{[\beta]_{\omega}}u_{\varepsilon}\to\lambda_{\beta}\in[0,\infty)$ for $3+2\omega\le\beta<3+3\omega$,*

the following asymptotical relation holds,

$$\exp\left\{(a_{1,1}\,\varepsilon^{1+\omega}+a_{2,2}\,\varepsilon^{2+2\omega})\,u_\varepsilon\right\}\Psi_\varepsilon(u_\varepsilon)$$
$$\to\exp\left\{-\lambda_\beta\,a_{3,2}\right\}\quad as\ \varepsilon\to 0.$$

(iv) *For any* $0\le u_\varepsilon\to\infty$ *in such a way that* $\varepsilon^{[\beta]_\omega}u_\varepsilon\to\lambda_\beta\in[0,\infty)$ *for* $3+3\omega\le\beta<4+3\omega$*,*

the following asymptotical relation holds,

$$\exp\left\{(a_{1,1}\,\varepsilon^{1+\omega}+a_{2,2}\,\varepsilon^{2+2\omega}+a_{3,2}\,\varepsilon^{3+2\omega})\,u_\varepsilon\right\}\Psi_\varepsilon(u_\varepsilon)$$
$$\to\exp\left\{-\lambda_\beta\,a_{3,3}\right\}\quad as\ \varepsilon\to 0.$$

(v) *For any* $0\le u_\varepsilon\to\infty$ *in such a way that* $\varepsilon^{[\beta]_\omega}u_\varepsilon\to\lambda_\beta\in[0,\infty)$ *for* $\beta=4+3\omega$*, the*

following asymptotical relation holds,

$$\exp\left\{(a_{1,1}\,\varepsilon^{1+\omega}+a_{2,2}\,\varepsilon^{2+2\omega}+a_{3,2}\,\varepsilon^{3+2\omega}+a_{3,3}\,\varepsilon^{3+3\omega})\,u_\varepsilon\right\}\Psi_\varepsilon(u_\varepsilon)$$
$$\to\exp\left\{-\lambda_\beta\,a_{4,3}\right\}\quad as\ \varepsilon\to 0.$$

Remark 5. This example of perturbed risk process was first introduced in the author's earlier paper (Ni, Silvestrov and Malyarenko 2008). Theorem 3 above is an extended version of Theorem 3 in the aforementioned paper. The latter theorem gives the corresponding result for the perturbation condition $\mathbf{P}_\omega^{(\xi)}$ for $\xi=3+2\omega$ and hence presents the expansion of ρ_ε only up to and including the term of order $O(\varepsilon^{3+2\omega})$. In Theorem 3 above, we determine two more terms for the expansion of ρ_ε and consequently obtain two additional variants, i.e. statements (iv) and (v), of the exponential asymptotics for the ruin probability. The proof of these additional results follows the same line as the proof of Theorem 3 in Ni, Silvestrov and Malyarenko (2008).

4.2. Perturbed Risk Process with Multivariate Non-polynomial Perturbations

Let us suppose that the claim size distribution $G_\varepsilon(z)$ for the risk process (8) is a mixture of exponential distributions of the following form

$$G_\varepsilon(z)\ =1-p_1e^{-z/\delta_1(\varepsilon)}-p_2e^{-z/\delta_2(\varepsilon)}-p_3e^{-z/\delta_3(\varepsilon)},\tag{19}$$

where

$\delta_i(\varepsilon)=\delta_i-C_i\varepsilon^{\omega_i}>0,\ \delta_i>0, C_i>0,\ i=1,2,3,$ for $\varepsilon\ge 0,\ 0\le p_1,p_2,p_3\le 1,$

$p_1+p_2+p_3=1$, $\omega_1\equiv 1$, and $\omega_2,\omega_3>1$ take irrational values such that ω_2/ω_3 is an irrational number. Without loss of generality we assume $\omega_2<\omega_3$, so that we can introduce the vector parameter $\vec{\omega}=(1,\omega_2,\omega_3)$.

The perturbation above can be seen as an environmental factor that determines claim amounts and acts in a different form for different claim groups.

Note that if the perturbation parameter $\varepsilon=0$, $G_\varepsilon(z)$ reduces to

$$G_0(z)=1-p_1e^{-z/\delta_1}-p_2e^{-z/\delta_2}-p_3e^{-z/\delta_3}.$$

The first moment of the perturbed claim size distribution, μ_ε, takes the form

$$\mu_\varepsilon = p_1(\delta_1 - C_1\varepsilon) + p_2(\delta_2 - C_2\varepsilon^{\omega_2}) + p_3(\delta_3 - C_3\varepsilon^{\omega_3}). \tag{20}$$

By (19) and (20), we have $G_\varepsilon(z) \to G_0(z)$ as $\varepsilon \to 0$ for every $z \geq 0$, and $\mu_\varepsilon \leq \mu_0$ but $\mu_\varepsilon \to \mu_0$ as $\varepsilon \to 0$.

Obviously, $\alpha_\varepsilon \leq \alpha_0 = 1$ and $\alpha_\varepsilon \to \alpha_0 = 1$ as $\varepsilon \to 0$, so we have again the case of diffusion approximation for ruin probabilities.

Since $\alpha_\varepsilon < \alpha_0 = 1$ for $\varepsilon > 0$, the distribution $F_\varepsilon(u) = \alpha_\varepsilon \widetilde{G}_\varepsilon(u)$ is improper for $\varepsilon > 0$ but the limiting function $F_0(u)$ is proper, i.e. the defect $f_\varepsilon = 1 - \alpha_\varepsilon \to f_0 = 0$ as $\varepsilon \to 0$.

It can be shown that the defect f_ε and the r-th moment $m_{\varepsilon r}$ for the distribution function $F_\varepsilon(u) = \alpha_\varepsilon \widetilde{G}_\varepsilon(u)$ take the following form:

$$f_\varepsilon = \frac{p_1}{\mu_0} C_1\varepsilon + \frac{p_2}{\mu_0} C_2\varepsilon^{\omega_2} + \frac{p_3}{\mu_0} C_3\varepsilon^{\omega_3}, \tag{21}$$

$$m_{\varepsilon r} = m_{0r} + \frac{r!}{\mu_0} \sum_{i=1}^{3} [p_i \sum_{j=1}^{r+1} \binom{r+1}{j} (\delta_i)^{r+1-j} (-C_i\varepsilon^{\omega_i})^j], r \geq 1. \tag{22}$$

Relations (21) and (22) imply that, in this case, the perturbation condition $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ holds for any $\alpha \geq 1$. It can also be shown that condition A, B and C hold for the perturbed renewal equation (10) with $G_\varepsilon(z)$ given by (19).

Instead of reformulating Theorem 2 for this case, we illustrate the asymptotic result by a specific example where $\omega_2 = \sqrt{2}, \omega_3 = \sqrt{3}$, i.e. $\vec{\omega} = (1, \sqrt{2}, \sqrt{3})$ and $\alpha = 3$. In this case the following exponential asymptotic expansion for the ruin probability can be obtained by applying Theorem 2.

THEOREM 4

Let the perturbed claim size distributions $G_\varepsilon(z)$ be given by formula (19) and ε be the perturbation parameter, let also condition ***D*** *holds. Then :*

(i) *There exists a unique non-negative solution, ρ_ε, of the characteristic equation, (3) and the following asymptotical relation holds*

$$\rho_\varepsilon = a_{(1,0,0)}\varepsilon + a_{(0,1,0)}\varepsilon^{\sqrt{2}} + a_{(0,0,1)}\varepsilon^{\sqrt{3}} + a_{(2,0,0)}\varepsilon^2 + a_{(1,1,0)}\varepsilon^{1+\sqrt{2}} + \\ + a_{(1,0,1)}\varepsilon^{1+\sqrt{3}} + a_{(0,2,0)}\varepsilon^{2\sqrt{2}} + a_{(3,0,0)}\varepsilon^3 + o(\varepsilon^3) \tag{23}$$

where $a_{(1,0,0)} \ldots a_{(3,0,0)}$ can be calculated using recurrent formula (6) with the use of formulas (21) and (22), in particular,

$$a_{(1,0,0)} = \frac{p_1 C_1}{\mu_0 m_{01}}, \quad a_{(0,1,0)} = \frac{p_2 C_2}{\mu_0 m_{01}}, \quad a_{(0,0,1)} = \frac{p_3 C_3}{\mu_0 m_{01}},$$

$$a_{(2,0,0)} = \frac{p_1^2 C_1^2 (4\delta_1 m_{01} - m_{02})}{2\mu_0^2 m_{01}^3}, \ldots$$

(ii) *For any* $0 \le u_\varepsilon \to \infty$ *in such a way that* $\varepsilon^{[\beta]_\omega} u_\varepsilon \to \lambda_\beta \in [0,\infty)$ *for some* $1 \le \beta < 3$*, the following asymptotical relations holds,*

$$\exp\{(\sum_{1 \le \vec{n}\cdot\vec{\omega} < [\beta]_{\vec{\omega}}} a_{\vec{n}}\, \varepsilon^{\vec{n}\cdot\vec{\omega}}) u_\varepsilon\} \Psi_\varepsilon(u_\varepsilon) \to \exp\{-\lambda_\beta a^{(1)}\} \quad as\ \varepsilon \to 0, \tag{24}$$

where $a^{(1)} = a_{\vec{p}}$ *with* $\vec{p} = \vec{f}(\beta, \vec{\omega})$.

Remark 6. The expansion for ρ_ε, (23) is expanded only up to order $O(\varepsilon^3)$ in the example above. If needed, ρ_ε can be further expanded up to order of $O(\varepsilon^{[\alpha]_{\vec{\omega}}})$ for any real number $1 \le \alpha < \infty$.

Remark 7. Although the above asymptotic results are derived for specific parameter values, i.e. $\omega_2 = \sqrt{2}$ and $\omega_3 = \sqrt{3}$, similar results can be easily obtained for cases where the parameters ω_2 and ω_3 take other admissible values. Different choices of ω_2 and ω_3 only lead to different forms of the expansion for ρ_ε.

Remark 8. As in Theorem 3, statement **(ii)** of Theorem 4 leads to several variants of asymptotic relation (24) for different cases of the values for β, namely $1 \le \beta < \sqrt{2}$, $\sqrt{2} \le \beta < \sqrt{3}$, $\sqrt{3} \le \beta < 2$, ..., $2\sqrt{2} \le \beta < 3$ and finally $\beta = 3$. For instance, under the the balancing condition described in statement **(ii)**, we have, if $1 \le \beta < \sqrt{2}$, the asymptotic relation:

$$\Psi_\varepsilon(u_\varepsilon) \to \exp\{-\lambda_\beta a_{(1,0,0)}\} \quad as\ \varepsilon \to 0. \tag{25}$$

Similarly if $\sqrt{2} \le \beta < \sqrt{3}$ we obtain

$$\exp\{(a_{(1,0,0)}\varepsilon) u_\varepsilon\} \Psi_\varepsilon(u_\varepsilon) \to \exp\{-\lambda_\beta a_{(0,1,0)}\} \quad as\ \ \varepsilon \to 0, \tag{26}$$

and following the same pattern, if $\sqrt{3} \le \beta < 2$ we obtain

$$\exp\{(a_{(1,0,0)}\varepsilon + a_{(0,1,0)}\varepsilon^{\omega_2}) u_\varepsilon\} \Psi_\varepsilon(u_\varepsilon) \to \exp\{-\lambda_\beta a_{(0,0,1)}\} \quad as\ \ \varepsilon \to 0,$$

and so on.

5. Experimental Study

In Section 5.1, we carry out experimental numerical studies for the example of perturbed risk process discussed in Section 4.1. The example introduced in Section 4.2 is investigated in Section 5.2.

5.1. Perturbed Risk Process with Bivariate Non-polynomial Perturbations

The asymptotic formulas given by statements **(i) - (v)** of Theorem 3 can serve as approximation methods for $\Psi_\varepsilon(u)$ for small value of ε and relatively large values of u. To gain insight into the accuracy and other properties of these asymptotic formulas, we compare the corresponding approximations to the value of ruin probability estimated by computer simulation since the true value is difficult to compute.

Let us denote the simulated estimate of $\Psi_\varepsilon(u)$ by $\Psi_\varepsilon^s(u)$ with s standing for simulation. To obtain $\Psi_\varepsilon^s(u)$ we implement the conditional Monte Carlo simulation method, i.e. a variance reduced version of the Crude Monte Carlo, via the Pollaczeck-Khinchine formula for ruin probabilities. The description of this simulation method can be found in Asmussen (2000). The solution to our problem is $\Psi_\varepsilon^s(u) = \mathrm{E}(Z)$ where Z is the random variable generated in Algorithm 1 below. Note that u is the chosen value of the initial reserve, T is the constant parameter of $G_\varepsilon(z)$ given in (14) and hence the corresponding constant in $\widetilde{G}_\varepsilon(u)$ defined in (11).

Algorithm 1

1. Generate geometric random variable K, with $P(K=k) = (1-\alpha_\varepsilon)\alpha_\varepsilon^k$.
2. **if** $K = 0$ **then** $Z \leftarrow 0$.
3. **else if** $K = 1$ **then** $Z \leftarrow 1 - \widetilde{G}_\varepsilon(u)$.
4. **else**
5. Generate $L_1, \ldots, L_{K-1}$ from distribution $\widetilde{G}_\varepsilon(\cdot)$.
6. let $Y \leftarrow u - (L_1 + \cdots + L_{K-1})$.
7. **end if**
8. **if** $Y < 0$ **then** $Z \leftarrow 1$. **else if** $Y > T$ **then** $Z \leftarrow 0$.
9. **else** $Z \leftarrow 1 - \widetilde{G}_\varepsilon(Y)$.
10. **end if**

The main problem in Algorithm 1 is to simulate the random variable from the distribution $\widetilde{G}_\varepsilon(\cdot)$. We use the inverse method to generate outcomes of this random variable, i.e. to generate

$$x = T_0 - [T_0^{\omega+1}(1-v) + \varepsilon^{\omega+1} v]^{\frac{1}{\omega+1}} \qquad 0 \le v \le 1, \tag{27}$$

where v is a realization of a standard uniform random variable.

Set the parameters $T_0 = 1$, $\omega = (4+\sqrt{2})/5$, the simulation experiments have been carried out for different combinations of initial capital u and the perturbation parameter ε, with concentration on the cases where the ruin probability is of the magnitude 10^{-2}, 10^{-3} and 10^{-5}. For each simulation experiment, we execute the block in Algorithm 1 for 100 million times, i.e., to generate 100 million replicates of random variable Z. When the ruin probability is as small as of magnitude 10^{-5}, we increase the number of simulations to 500 million times.

Let us denote $\Psi_\varepsilon^j(u), j = 1,\ldots,5$ as the approximated ruin probability via the j-th statement in Theorem 3. By inspecting statement **(i) - (v)** we note that $\Psi_\varepsilon^j(u)$ represents the approximation where the j-th order expansion of ρ_ε is used in the corresponding asymptotic formula. Let us call $\Psi_\varepsilon^j(u)$ the j-th order approximation.

The relative errors $E_j(u,\varepsilon),\ j=,1\ldots,5$ are calculated in the traditional way as

$$E_j(u,\varepsilon)=\frac{\Psi_\varepsilon^j(u)-\Psi_\varepsilon^s(u)}{\Psi_\varepsilon^s(u)},$$

and they are presented in Table 1. The value of safety loadings, η_ε, are also given in the table.

Table 1. Relative errors of the approximation by Theorem 3 with $T_0=1, \omega=(4+\sqrt{2})/5$

ε (η_ε)	u	$\Psi_\varepsilon^s(u)$	Relative errors $E_j(u,\varepsilon)$ (%) E_1	E_2	E_3	E_4	E_5
0.05 (0.2%)	500	0.0487	1.51	0.13	0.19	0.18	0.18
	800	0.0080	2.42	0.21	0.31	0.29	0.29
	1600	6.35×10^{-5}	4.37	-0.10	0.10	0.07	0.07
0.1 (0.8%)	100	0.0743	5.3	0.27	0.71	0.57	0.59
	200	0.0055	10.4	0.13	1.01	0.74	0.79
	400	3.121×10^{-5}	20.3	-1.11	0.64	0.08	0.18
0.15 (2.0%)	50	0.0450	14.6	0.41	2.21	1.32	1.57
	100	0.0021	29.7	-0.55	3.05	1.27	1.76
	170	2.69×10^{-5}	56.1	-0.58	5.62	2.54	3.37
0.2 (3.6%)	30	0.0301	30.6	0.28	5.13	1.87	3.05
	50	0.0030	52.9	-1.48	6.59	1.13	3.09
	90	2.86×10^{-5}	111.8	-4.03	10.58	0.59	4.13
0.3 (8.9%)	10	0.0492	65.1	2.54	16.52	2.06	9.75
	20	0.0026	155.7	-1.36	27.38	-2.28	13.01
	35	3.08×10^{-5}	394.0	-6.69	45.95	-8.22	18.37
0.35 (12.7%)	8	0.0290	115.7	4.60	31.22	-0.61	18.73
	15	0.0014	292.7	1.06	54.59	-8.18	28.15
	24	2.91×10^{-5}	747.5	-3.41	90.68	-17.14	41.25

The first impression of Table 1 is $E_1(u,\varepsilon)$ is far too large when $\varepsilon\geq0.1$ for all chosen values of u. This suggests that the first order approximation $\Psi_\varepsilon^1(u)$ is not adequate unless ε is really small, thus the contribution of the second term in (18) is definitely not negligible. For $\varepsilon\leq0.2$, the higher order approximations, namely $\Psi_\varepsilon^j(u), j=2,\ldots,5$ are good, except that the approximation by $\Psi_\varepsilon^3(u)$ does not work so well for $\varepsilon=0.2$, which may be caused by some special property of the expansion (18).

As seen from Table 1, if ε is relatively large, say $\varepsilon\geq0.2$, even higher order approximations work poorly. Interestingly, in general approximation by $\Psi_\varepsilon^2(u)$ still seem to be applicable.

These experiments are done for T_0 normalized to 1, and ω set to equal to $(4+\sqrt{2})/5$. Similar experiments have been done for $T_0 = 1, \omega = \sqrt{2}$, and the general impression of the results is the same: first order approximation ought not to be used for moderate and large ε; all approximations get more accurate as ε gets smaller as expected. The quality of approximations seems to depend heavily on ε but not so much on the values of u in the chosen range.

It can be shown that in this model of a perturbed risk process, the safety loading η_ε is of the order $O(\varepsilon^{1+\omega})$. Therefore as shown in Table 1, the approximations are applicable only for η_ε being very small. This may not be the most interesting case in risk theory. However, we would like to note that there's a duality of the classical risk process with the virtual waiting time process in a M/G/1 queue, consequently the ruin probability can be interpreted as the steady-state limit of the virtual waiting time. The case when η_ε is very small corresponds to the interesting heavy traffic case in the queuing theory and thus the study of this case has its own value.

Finally, we compare the approximation by $\Psi_\varepsilon^2(u)$ to the classical diffusion approximation method (see for example Grandell 2000),

$$\Psi_\varepsilon^D(u) = \exp(-(2\beta_\varepsilon \eta_\varepsilon u)/\gamma_\varepsilon), \tag{28}$$

where $\beta_\varepsilon, \gamma_\varepsilon$ refer to the first and second moment of claim size distribution $G_\varepsilon(z)$, η_ε is the safety loading. The results are presented in Table 2, with $E_D(u,\varepsilon)$ refer to the relative error of the approximation by (28).

As seen from Table 2, in this numerical example, approximation by $\Psi_\varepsilon^2(u)$ works better. $E_D(u,\varepsilon)$ tends to get larger as ε get larger and also as u gets larger. This is the case for $\varepsilon > 0.2$ as well (not shown in the table).

5.2. Perturbed Risk Process with Multivariate Non-polynomial Perturbations

We consider a numerical example for the application in section (4.2). Suppose that $C_1 = C_2 = C_3 = 1$, $p_1 = 0.4, p_2 = 0.3, p_3 = 0.3,\ \ \delta_1 = 3, \delta_2 = 5, \delta_3 = 7$ in the perturbed claim size distribution (19).

Since the claim distribution $G_\varepsilon(z)$ is a mixture of three exponential distributions, exact formula of ruin probability for this case exists in terms of a matrix-exponential function (see for example Asmussen 2000). Let us denote the ruin probability calculated via the exact formula by $\Psi_\varepsilon^e(u)$. We then compare it to the approximated ruin probabilities via statement **(ii)** of Theorem 4. Let us denote $\Psi_\varepsilon^j(u), j = 1, \ldots 8$ as these approximated ruin probabilities with β chosen in such a way that the j-th order expansion of ρ_ε is used in (24). For example, $\Psi_\varepsilon^1(u)$, call it the first order approximation, is calculated using (25) in Remark 8 where the parameter β satisfies $1 \le \beta < \sqrt{2}$, and $\Psi_\varepsilon^2(u)$, i.e. the second order approximation, refers to the approximation by (26) and so on.

Table 2. Relative errors of $\Psi_\varepsilon^j(u)$ and $\Psi_\varepsilon^D(u)$

$\varepsilon\,(\eta_\varepsilon)$	u	$\Psi_\varepsilon^s(u)$	$E_2(u,\varepsilon)(\%)$	$E_D(u,\varepsilon)(\%)$
0.05 (0.2%)	500	0.0487	0.13	0.26
	800	0.0080	0.21	0.42
	1600	6.35×10^{-5}	-0.10	1.34
0.1 (0.8%)	100	0.0743	0.27	1.03
	200	0.0055	0.13	2.43
	400	3.121×10^{-5}	-1.11	6.12
0.15 (2.0%)	50	0.0450	0.41	2.96
	100	0.0021	-0.55	7.11
	170	2.69×10^{-5}	-0.58	11.47
0.2 (3.6%)	30	0.0301	0.28	6.44
	50	0.0030	-1.48	12.25
	90	2.86×10^{-5}	-4.03	22.08

The relative errors $E_j(u,\varepsilon),\ j=1,\ldots,8$, defined as

$$E_j(u,\varepsilon)=\frac{\Psi_\varepsilon^j(u)-\Psi_\varepsilon^e(u)}{\Psi_\varepsilon^e(u)},$$

are calculated for different combinations of ε and u and presented in Table 3. All calculations are done in MATLAB. Symbol η_ε in Table 3 refers to the safety loading coefficient.

As shown in Table 3, the higher order approximations, i.e. $\Psi_\varepsilon^j(u), j\ge 3$ are perfect for ε small, say $\varepsilon\le 0.05$. For $0.05<\varepsilon\le 0.15$, even higher order approximations $\Psi_\varepsilon^j(u), j\ge 6$ should be used. Also it is seen from the table that, for a fixed u and a fixed small ε, the relative errors appear to decrease, when we include more terms from the expansion ρ_ε in the approximation, with the exception that $E_8(u,\varepsilon)$ is oftentimes slightly larger than $E_7(u,\varepsilon)$.

Figure 1 illustrates how the approximation in general improves as we take approximations of higher orders. Approximation $\Psi_\varepsilon^1(u)$ is shown to be a very poor approximation and is therefore omitted in the figure.

Table 3. Relative errors of the approximation by Theorem 4 with $C_1=C_2=C_3=1$, $p_1=0.4, p_2=0.3, p_3=0.3,\ \ \delta_1=3,\delta_2=5,\delta_3=7$

ε (η_ε)	u	$\Psi_\varepsilon^e(u)$	Relative errors $E_j(u,\varepsilon)$ (%)							
			E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
0.01 (0.0009)	17000	0.0499	44	7.1	0.14	0.13	0.12	0.11	0.10	0.11
	33000	0.0030	102	14	0.18	0.16	0.13	0.11	0.11	0.11
	57000	4.31×10^{-5}	237	26	0.23	0.21	0.14	0.11	0.11	0.11
0.05 (0.005)	3000	0.0481	103	22	0.99	0.96	0.79	0.67	0.63	0.64
	6000	0.0023	310	50	1.48	1.33	0.99	0.74	0.67	0.69
	10000	4.11×10^{-5}	947	95	1.91	1.83	1.26	0.84	0.72	0.76
0.15 (0.020)	1000	0.0273	258	62	4.65	4.57	3.75	2.88	2.61	2.71
	1500	0.0046	570		5.93	5.81	4.56	3.26	2.85	3.00
	3000	2.12×10^{-5}	4297	305	9.86	9.61	7.04	4.38	3.57	3.86
0.25 (0.037)	500	0.0343	320	85	9.34	9.23	7.75	5.95	5.36	5.59
	800	0.0046	870	162	12.7	12.5	10.04	7.11	6.16	6.53
	1500	4.37×10^{-5}	6735	487	20.8	20.4	15.6	9.86	8.05	8.75

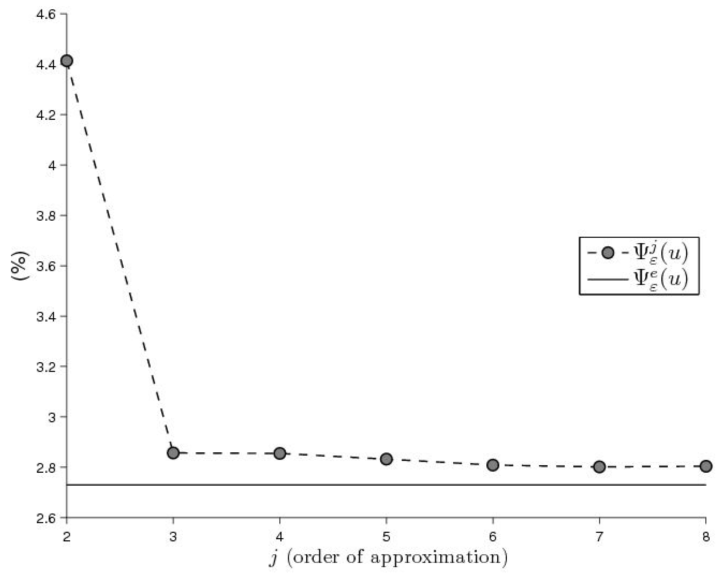

Figure 1. Approximation by $\Psi_\varepsilon^j(u)$ for $\varepsilon = 0.15; u = 1000.$

From Table 3 we note also that, for a fixed $\varepsilon \geq 0.05$ and a fixed approximation, if u takes a larger value, the corresponding relative error appears to be larger. This seems to be contradictory to the fact that formula (24) holds for $u \to \infty$ and $\varepsilon \to 0$ simultaneously. However, note that to use formula (24) we should have $u \to \infty$ and $\varepsilon \to 0$ balanced so that $\varepsilon^{[\beta]\omega} u_\varepsilon \to \lambda_\beta \in [0, \infty)$ for some $1 \leq \beta < 3$. Hence the value of λ_β can have a subtle effect on the quality of approximation. The experiments suggest a relatively too large λ_β may not be desirable for a good approximation. For the purpose of illustration, let us consider the approximation by $\Psi_\varepsilon^3(u)$ with $\varepsilon = 0.15$ and varying u, of which the values of λ_β are given in Table 4:

Table 4. The values of λ_β for $\Psi_\varepsilon^3(u)$ with $\varepsilon = 0.15$ and varying u

ε	u	$\Psi_\varepsilon^e(u)$	$E_3(u,\varepsilon)$ (%)	λ_β
0.15	1000	0.0273	4.65	37.4
	1500	0.0046	5.93	56.1
	3000	2.12 $\times 10^{-5}$	9.86	112.2

We note from Table 4 that when λ_β is as large as 112.2, the approximation is less accurate for the cases with smaller values of λ_β. To address questions like whether the values of λ_β always affect the quality of approximation, and if this is true which value of λ_β is optimal for the approximation, more comprehensive and extensive numerical experiments are required.

6. Conclusions and Future Research

We have studied the asymptotic behavior of nonlinearly perturbed equations with non-polynomial perturbations of the type $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ which is a generalized type of the non-polynomial perturbations treated in the previous research (Ni, Silvestrov, Malyarenko 2008). The theoretical results have been applied to examples of nonlinearly perturbed risk processes and can have potential applications in various applied probability models. For the proofs of the results we refer to a forthcoming report by Ni (2010).

This article has dealt with asymptotically proper perturbed renewal equation, i.e. as described in condition A, $F_0(t)$ is assumed to be a proper distribution function. The case of asymptotically improper perturbed renewal equation where $F_0(t)$ can be improper leads to a further generalization of the theory and will be studied at the next stage of research. The study of asymptotic expansions for renewal limits follows naturally afterwards.

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