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FUZZY PROBABILISTIC MODELS FOR STRUCTURAL SERVICEABILITY[1]

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Abstract: *Structural serviceability is of uttermost importance for the overall performance of many common structures. As a rule, both the load effects (serviceability indicators due to loading) and admissible constraints (ensuring required structural performance) are random variables of considerable scatter and significant vagueness. Common experience indicates that a structure does not loose its ability to comply with specified performance requirements abruptly at a distinct point of the serviceability indicator, but gradually within a certain transition interval. Fuzzy-probabilistic methods are therefore employed to analyze the structural serviceability.*
As an example, serviceability limit states of water retaining structures with respect to cracking are investigated in detail. Fuzzy probabilistic models are proposed to derive theoretical models for the limiting crack width. It is shown that the fuzzy probabilistic distribution of serviceability requirements may be used similarly as classical distribution function to specify the characteristic value of limiting crack width, to analyze reliability of crack width and to optimize structural design to achieve the minimum total costs.

Key words*: fuzzy probabilistic models; probability; serviceability; cracks width; optimization*

1. Introduction

Structural performance is becoming a fundamental concept of advanced engineering design in construction. However, performance requirements (including serviceability, safety, security, comfort, functionality) of buildings and engineering works are often affected by various uncertainties that can hardly be entirely described by traditional probabilistic models. As a rule, transformation of human desires, particularly those describing occupancy comfort and aesthetic aspects, to performance (user) requirements often results in an indistinct or imprecise definition of the technical criteria for relevant performance indicators (for example the limiting defection or crack width).

Thus, in addition to natural randomness of basic variables, performance requirements may be considerably affected by vagueness in the definition of technical criteria. Two types of uncertainty of performance requirements are therefore identified here: randomness, handled by commonly used methods of the theory of probability, and, fuzziness, described by basic tools of the recently developed theory of fuzzy sets (Brown and Yao 1983). Similarly as in the previous studies (Holický 2006), the fundamental condition of structural performance, $S \le R$, between an action effect S and a relevant performance requirement R, is considered assuming the randomness of S and both the randomness and fuzziness of R. In this study the performance resistance R is analysed in detail.

An illustrative example of continuous vibration in offices is used throughout the paper to clarify general concepts. In this example, it is shown that it is impossible to identify a distinct value of an appropriate indicator (root mean square value of acceleration) that would separate a satisfactory from an unsatisfactory performance (Holický et al. 2009). Typically, a broad transition region is observed, where the building is gradually losing its ability to perform adequately and where the degree of damage (inadequate performance or malfunction) gradually increases.

2. Fuzzy Probabilistic Models of Performance Requirements

Fuzziness due to vagueness and imprecision in the definition of performance requirement R is described by the membership function $v_R(x)$ indicating the degree of the membership of a structure in a fuzzy set of damaged (unserviceable) structures (Holický 2006); here x denotes a generic point of a relevant performance indicator (a deflection or a root mean square of acceleration) considered when assessing structural performance. Common experience indicates that a structure is loosing its ability to comply with specified requirements gradually within a certain transition interval $\langle r_1, r_2 \rangle$.

The membership function $v_R(x)$ describes the degree of structural damage (lack of functionality). If the rate $dv_R(x)/dx$ of the "performance damage" in the interval $\langle r_1, r_2 \rangle$ is constant (a conceivable assumption), then the membership function $v_R(x)$ has a piecewise linear form as shown in Figure 1. It should be emphasized that $v_R(x)$ describes the non-random (deterministic) part of uncertainty in the requirement R related to economic and other consequences of inadequate performance. The randomness of R at any damage level $v = v_R(x)$ may be described by the probability density function $\varphi_R(x|v)$ (see Figure 1), for which a normal distribution having the constant coefficient of variation $V_R = 0.10$ is considered in the following.

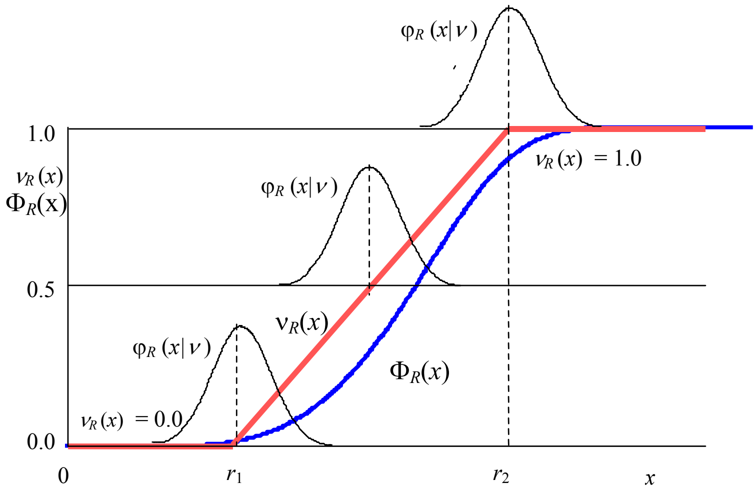

Figure 1. The fuzzy probabilistic model of the performance requirement R

The transition region $\langle r_1, r_2 \rangle$, where the structure is gradually losing its ability to perform adequately and its damage increases, may be rather broad, depending on the nature of the performance requirement. For common serviceability requirements (deflections) the upper limit r_2 may be a multiple of the lower limit r_1 (for example, $r_2 = 2 \cdot r_1$).

The fuzzy probabilistic measures of structural performance is defined as the damage function $\Phi_R(x)$ being the weighted average of damage probabilities reduced by the corresponding damage level (Holický 2006)

$$\Phi_R(x) = \frac{1}{N}\int_0^1 \nu\left(\int_{-\infty}^{x} \varphi_R(x' \mid \nu)\mathrm{d}x'\right)\mathrm{d}\nu\,, \qquad (1)$$

where N denotes a factor normalizing the damage function $\Phi_R(x)$ to the conventional interval $\langle 0,1 \rangle$ (see Figure 1) and x' is a generic point of x. The density of the damage $\varphi_R(x)$ follows from (1) as

$$\varphi_R(x) = \frac{1}{N}\int_0^1 \nu\ \varphi_R(x \mid \nu)\ \mathrm{d}\nu\,. \qquad (2)$$

The damage function $\Phi_R(x)$ and density function $\varphi_R(x)$ defined by equation (1) and (2) may be considered as generalized distribution functions of the performance requirements R that can be used similarly as classical probabilistic functions.

3. Fuzzy Probability of Performance Failure

The damage function $\Phi_R(x)$ defined by equation (1) may be used similarly as the classical distribution function of structural resistance. If the action effect S of a structural member is well-known and its probability density function $\varphi_S(x)$ is available, the fuzzy probability of performance failure π_f may be assessed as

$$\pi_f = \int_{-\infty}^{\infty} \varphi_S(x)\Phi_R(x)\mathrm{d}x\,. \tag{3}$$

The damage function $\Phi_R(x)$ defined by equation (1) and the fuzzy probability of performance failure π defined by equation (3) enable the formulation of various design criteria in terms of relevant randomness and fuzziness parameters. In addition, fuzzy probabilistic optimization can be used to specify the optimum structural design and appropriate fuzzy reliability level. However, adequate data for the specification of the fuzziness parameters r_1, r_2, the membership function $v_R(x)$ and its coefficient of variation V_R (describing the requirement R) and the probability density $\varphi_S(x)$ of the load effect S are needed.

4. The Characteristic Value of Performance Requirement

The characteristic value r_K of the performance requirement R can be determined as a specified fractile of the damage function $\Phi_R(x)$

$$\pi_k = \Phi_R(r_k)\,. \tag{4}$$

Here π_k is the fuzzy probability of not achieving the characteristic value r_K. It may differ from the commonly accepted value $\pi_k = 0.05$ in the case of classical definition of probability. Previous studies (Holický 2006) based on the fuzzy probabilistic optimization indicate that the characteristic value of serviceability requirements (limiting value in design) corresponding to the probability $\pi_k = 0.05$ may not lead to the optimum reliability level.

5. The Limiting Crack Width for Water Retaining Structures

Water retaining structures are usually designed on the basis of crack width requirements. The limiting values are commonly within the interval from $r_1 = 0.05\,mm$ to $r_2 = 0.2\,mm$ (Holický et al. 2009). Considering these values as the deterministic lower and upper bounds of the transition region, the membership function $v_R(x)$, the damage function $\Phi_R(x)$ and the density function $\varphi_R(x)$ defined by equations (1) and (2) are shown in Figure 2. It should be mentioned that the transition region might be slightly shifted if the values $r_1 = 0.05\,mm$ and $r_2 = 0.2\,mm$ are considered as fractiles of the admissible crack width. Figure 2 also indicates the characteristic value of the limiting crack width $r_k = 0.082\,mm$ corresponding to the conventionally accepted probability $\pi_k = 0.05$ of not achieving r_k, if the probability $\pi_k = 0.20$, then the characteristic value is $r_k = 0.115\,mm$.

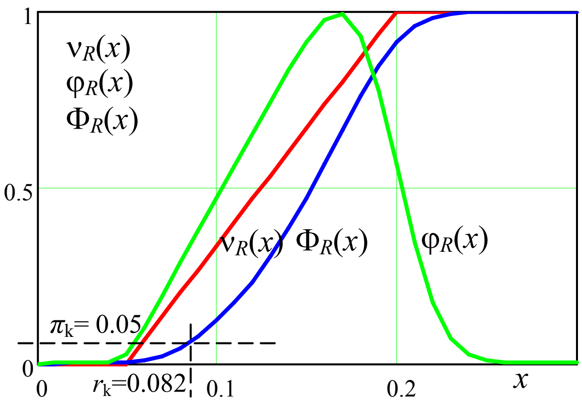

Figure 2. The membership function $\nu_R(x)$, the damage function $\Phi_R(x)$ and the damage density function $\varphi_R(x)$ for the transition region from $r_1 = 0.05\,mm$ to $r_2 = 0.2\,mm$ and $V_R = 0.10$

It follows from Figure 2 that the characteristic value $r_k = 0.082\,mm$ is relatively close to the lower bound of transition region $a = 0.05\,mm$. However, as indicated above, in the case of serviceability requirements the probability $p = 0.05$ may not be the optimum value used to define the characteristic serviceability resistance, for example the limiting crack width.

Note that damage density may be well approximated by the Beta distribution having the mean $0.15\,mm$, standard deviation $0.039\,mm$, the lower bound $0.02\,mm$ and the upper bound $0.238\,mm$.

6. Fundamental Concepts in Eurocodes

6.1. Crack Width

Verification of cracking is mostly based on semi empirical formulae supported by experimental evidence, experience and structural detailing (EN 1992-1-1 (2004), Narayanan and Beeby (2005)). A number of different approaches leading to considerably diverse results may be found in literature and codes of practice ((EN 1992-1-1 (2004), EN 1992-1-3 (2006), Narayanan and Beeby (2005)). The following probabilistic study is based on the concepts accepted in Eurocodes. Basic relationship for the assessment of crack width w is written in the form of simple compatibility condition (EN 1992-1-1 (2004), Narayanan and Beeby (2005))

$$w_m = S_{rm}\varepsilon_m, \qquad (5)$$

where w_m is the mean crack width, S_{rm} the mean crack spacing and ε_m the mean strain in between the two adjacent cracks. The mean crack spacing S_{rm} can be assessed using a semi empirical formula (EN 1992-1-1 (2004), Narayanan and Beeby (2005))

$$S_{rm} = 2c + 0.25 \cdot k_1 \cdot k_2 \cdot \phi / \rho_{eff} \,, \tag{6}$$

where c denotes concrete cover, k_1 is a coefficient taking into account bond properties of the reinforcement (a value 0.8 for high bond and 1.6 for smooth bars), k_2 is a coefficient depending on the form of stress distribution (a value 0.5 for bending, 1.0 for pure tension), ϕ is the bar diameter and ρ_{eff} the effective reinforcement ratio $A_s / A_{c,eff}$. Here A_s is the reinforcement area and $A_{c,eff}$ is the effective concrete area surrounding the reinforcing bars. Detailed instructions on how to determine the area $A_{c,eff}$ are provided in EN 1992-1-1 (2004). Note that $A_{c,eff}$ is usually smaller than the concrete area A_c considered normally for the reinforcement ratio of flexural or compressive members, and, consequently, the effective reinforcement ratio ρ_{eff} may be greater than the commonly used reinforcement ratio ρ.

The mean strain ε_m for reinforced concrete members (non prestressed) may be calculated from the expression (EN 1992-1-1 (2004), Narayanan and Beeby (2005))

$$\varepsilon_m = \varepsilon_{sm} - \varepsilon_{cm} = \frac{\sigma_s - k_t f_{ct,eff}(1 + \alpha_e \rho_{eff}) / \rho_{eff}}{E_s} \geq 0.6 \frac{\sigma_s}{E_s} \,, \tag{7}$$

where ε_{sm} is the mean strain in reinforcing bars, ε_{cm} the mean strain in surrounding concrete, σ_s is the stress in tension reinforcement at the crack section, k_t is a factor dependent on the duration of the load (0.6 for short term loading, 0.4 for long term loading), $f_{ct,eff}$ is the mean of the tensile strength of the concrete, effective at the time when the cracks may first develop ($f_{ct,eff} \leq f_{ctm}$), and α_e is the ratio modulus E_s / E_{cm}.

6.2. Design Condition

To verify the mean crack width, w_m is multiplied by the factor $\beta_w (= 1.7)$ and compared with the limiting crack width w_d. Thus, it is required that

$$w_k \approx \beta_w w_m < w_{\lim} \,. \tag{8}$$

It is assumed that the product $w_k \approx \beta_w w_m$ is called the characteristic value of the crack width, which is supposed to be equal to the upper 5% fractile of the crack width w. The required value $w_{\lim}$ is considered as a deterministic quantity (for water retaining structures up to $0.2\,mm$).

6.3. Load Combinations

The quasi-permanent combinations of actions are usually considered in design verification of crack width as follows (EN 1990 (2002)):

$$G_k + L_k + \psi Q_k \,. \tag{9}$$

Here G_k denotes the characteristic value of the permanent load G, L_k stands for the characteristic value of the liquid load L (considered similarly as the permanent load, $\mu_L = L_k$), Q_k is the characteristic value of the variable load Q, ψ is the combination factor for the variable load Q. In some cases (for example in case of a wall of water retaining structures) the liquid load L can be considered only (effect of other loads are negligible). In the design verification of ultimate limit states the partial factors for all actions should be considered as recommended in relevant codes, for the liquid load L the partial factor should be considered as $\gamma = 1.2$ as recommended in EN 1992-3 (2006).

7. Probabilistic Formulation

7.1. The Limit State Function

Random behavior of crack width w can be analyzed using equations (5), (6) and (7), where all input quantities are considered as random variables. Equation (5) can be thus written as

$$w = S_r \varepsilon . \qquad (10)$$

Here w denotes the crack width, S_r is the crack spacing and ε is the strain as random variable. The crack spacing S_r is assumed to be described by equation (6), the strain ε by equation (7) assuming that all input quantities are considered as random variables having the means equal to nominal values. In equation (7) the lower bound $0.6 \cdot \sigma_s / E_s$ is not considered in the following reliability analysis.

The theoretical model for the strain ε is partly based on experimental observation. Its uncertainty is taken into account by a factor θ expressing model uncertainty. The limit state function g may be then written in a simple form

$$g = w_{\lim} - \theta w . \qquad (11)$$

Here the random crack width is given by equation (10), (6) and (7). The model uncertainty θ is introduced as an additional random variable (having the mean equal to unity and the coefficient of variation 10%). In the following analysis the limiting crack width $w_{\lim}$ is considered as a fuzzy random serviceability resistance R analysed above. It is defined by general equation (1) and described by the damage function (2) or the damage density function (3). A particular form of these functions relevant to the foreseen example of water retaining structures is shown in Figure 2.

7.2. Theoretical Models of Basic Variables

All the quantities entering equations (6), (7) and (11) including the model uncertainty θ are in general random variables. Some of them are, however, approximated by deterministic values (those having relatively small variability). Theoretical models (including the type of distribution and their parameters) of all variables used in the following reliability analysis are indicated in Table 1.

Table 1. Theoretical models of basic variables

Name	Symbol X	Unit	Distribution	Char. Value X_k	The mean μ_X	St. dev. σ_X
Width	b	m	Det	1.00	1.00	0
Cover	c	m	Gamma	0.04	0.04	0.01
Reinf. diam.	ϕ	m	Det	0.012 to 0.03	0.012 to 0.03	0
Tensile strength	f_{ct}	MPa	LN	2.9	2.9	0.55
Steel mod.	E_s	GPa	Det	200	200	0
Concrete mod.	E_c	GPa	Det	33	33	0
Creep coeffic.	φ	-	Det	2	2	0
Coefficient	k_1	-	Det	0.8	0.8	0
Coefficient	k_2	-	Det	1	1	0
Coefficient	k_t	-	Det	0.4	0.4	0
Limiting width	w_{lim}	m	Beta*)	0.0000823	0.00015	0.000039
Pressure	L_k	MPa	N	0.07	0.07	0.0035
Diameter	D	m	Det	28	28	0
Action uncer.	θ_E	-	LN	1.00	1.00	0.10

*) Parameters of the Beta distribution are derived from the above fuzzy probabilistic analysis of the limiting crack widths considering the lower limit value of the transition region $0.05\,mm$ and the upper limit $0.2\,mm$. The lower bound of Beta distribution $a = 0.02\,mm$ and the upper bound $b = 0.238\,mm$

The following notations are used in Table 1: Normal - for normal distribution, Gamma - for gamma distribution, LN - for log-normal distribution, Det - for deterministic value. Note that the model uncertainty θ is supposed to cover uncertainties in some variables that are indicated as deterministic quantities.

It follows from Table 1 that the limiting crack width $w_{\lim}$ is approximated by Beta distribution indicated in the above general analysis of fuzzy random properties of the serviceability resistance R.

8. Reliability Analysis

8.1. An Example of Water Reservoir

As an example of probabilistic design for cracking a cylindrical water reservoir with diameter $D = 28\,m$, height $7\,m$ (the maximum water pressure $L_k = 70\,kN/m^2$) and wall thickness $0.25\,m$ is considered (Holický et al. 2009). Crack width is analyzed in the wall only under pure tension due to water pressure. The maximum characteristic force in the wall is thus

$$N_s = D \cdot L_k / 2 = 980\,kN\,. \tag{12}$$

The basic reinforcement area $A_0 = 0.0027\,m^2$ in the wall is determined considering the ultimate limit state of tensile capacity of the wall using the partial load factor $\gamma = 1.2$, thus the design force in the wall is $N_d = \gamma N_s = 1{,}176\,kN$.

It is common that the basic reinforcement A_0 must be increased to an acceptable value A in order to control cracking. For the data given in Table 1 the enhancement factors given by ratio A/A_0 follow from general equations (5) to (7). For the deterministic design for crack width control according to EN 1992-1-1 (2003) the enhancement of the deterministic crack limit $w_{\lim} = 0.20$ is more than a factor of 2, and for the crack limit $w_{\lim} = 0.05$ it is more than 5, depending on the steel diameter. In the following analysis these outcomes of the deterministic calculation are compared with results of probabilistic analysis.

8.2 Probabilistic Analysis

Crack width of the reservoir wall exposed to pure tension is analyzed considering the limit state function (11) and theoretical models of basic variables given in Table 1. Various diameters of the reinforcing bars ϕ (from 12 to $30\,mm$) are considered. The limiting crack width $w_{\lim}$ is generally described by the Beta distribution. Note, however, that this approximation is derived from deterministic limiting crack widths $w_{\lim} = 0.05\,mm$ and $w_{\lim} = 0.20\,mm$. Figure 3 shows the variation of the failure probability with increasing area A/A_0 within the broad range from 1 to 5.

It follows from Figure 3 that without substantial enhancement of the reinforcement area the crack width would exceed the required limiting width $w_{\lim}$ with a very high probability. The basic reinforcement area A_0 should be increased approximately by the factor of 2 to comply with the required crack width. Figure 3 also indicates the fuzzy probability of failure $\pi_f = 0.05$ accepted in EN 1992-1-1 (2004) for verification of serviceability limit states including crack widths.

Desired enhancement of the reinforcement depends obviously on the reinforcing bars´ diameter ϕ. Figure 3 shows that for $\phi = 12\,mm$ the reinforcement ratio A/A_0 should be about 2.3, for $\phi = 30\,mm$ the reinforcement ratio A/A_0 should be about 3.2. This finding induces a crucial question concerning the required reliability level. In some cases the reliability may be decreased (target failure probability increased), in other cases (for example in case of a vital reservoir) it may be increased (target failure probability decreased). It appears that the methods of probabilistic optimization may provide a valuable guidance.

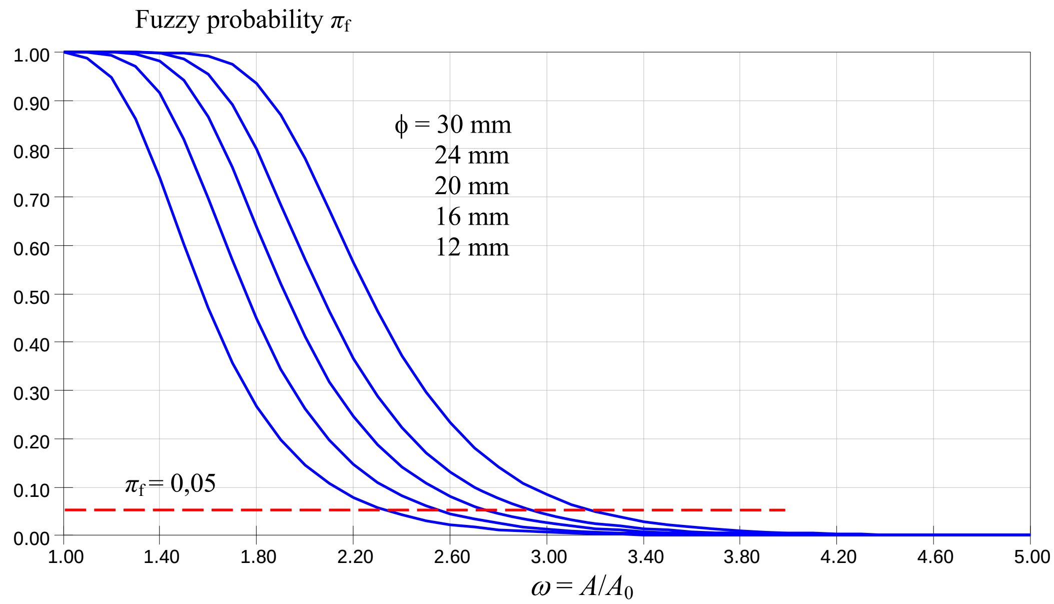

Figure 3. Variation of the probability of failure with the reinforcement area ratio $\omega = A/A_0$ for selected reinforcement diameter ϕ

9. Probabilistic Optimization

Probabilistic optimization may be effectively used to specify the optimum value of some basic variables (decisive parameters) and the target reliability of a structure. In some cases of the design of a concrete structure for cracking the objective function may be written in a simple form as the total cost

$$C_{tot}(\omega) = C_0 + C_1\omega + C_f \pi_f(\omega), \tag{13}$$

where $C_{tot}(\omega)$ denotes the total cost,

C_0 - the initial cost,

C_1 - the margin cost per unit of the decisive parameter ω,

C_f - the discounted cost serviceability failure,

$\pi_f(\omega)$ - fuzzy probability of failure,

ω - the decision parameter.

Here the initial cost C_0 is assumed to be independent of parameter ω. The product $C_1\omega$ denotes the additional cost due to an increase of parameter ω and $C_f\pi_f(\omega)$ the expected malfunctioning cost. The discounted cost of serviceability failure C_f takes into account the time when the crack width w exceeds the limit value $w_{\lim}$. The probability of failure $\pi_f(\omega)$ is considered as a function of the parameter ω. Instead of the total cost $C_{tot}(\omega)$ given by equation (13) the normalized $\kappa_{tot}(\omega)$ may be considered

$$\kappa_{tot}(\omega) = [C_{tot}(\omega) - C_0]/C_1 = \omega + \pi_f(\omega)C_f/C_1 \,. \tag{14}$$

It follows from the first derivative of $\kappa_{tot}(\omega)$ that the necessary condition for the optimum parameter ω_{opt} can be written as

$$\partial P_f(\omega)/\partial\omega = -C_1/C_f \,. \tag{15}$$

In the design of a concrete structure for cracking the reinforcement area A is optimized. A generic value of A may be introduced through the reinforcement ratio $\omega = A/A_0$, where the basic value A_0 is given by the ultimate limit state design. The optimum value $\omega_{opt} = A_{opt}/A_0$ can be assessed from the minimum of the standardized cost $\kappa_{tot}(\omega)$ given by equation (14) or directly from the necessary condition (15).

Figure 4 shows the variation of the reliability index β and the total standardized costs $\kappa_{tot}(\omega)$ given by equation (14) with the reinforcement area ratio $\omega = A_s/A_{s0}$. It follows from Figure 4 that the optimum parameter ω_{opt} increases with the cost ratio C_f/C_1; for $C_f/C_1 = 1$, $\omega_{opt} = 1.0$, for $C_f/C_1 = 1{,}000$, $\omega_{opt} = 4.3$. Thus, in general, the reinforcement area A needs to be substantially increased to reach the minimum total cost $\kappa_{tot}(\omega)$.

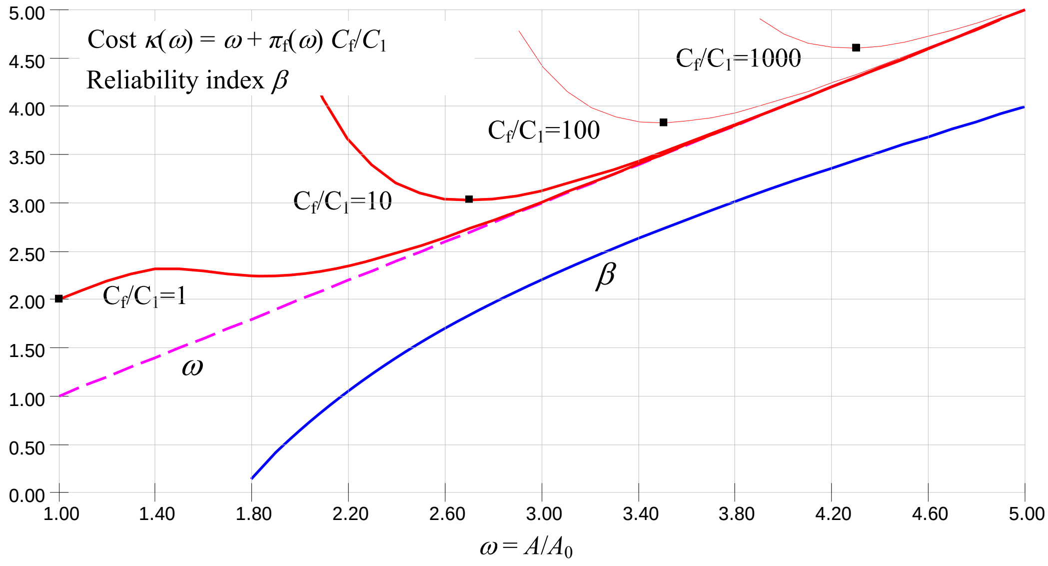

Figure 4. Variation of the total normalized cost $\kappa(\omega)$ and reliability index β with the reinforcement ratio ω for the reinforcement diameter $\phi = 16\,mm$

Figure 4 also indicates that the reliability index β is within a broad interval from 0 to 3.5. Thus, for high costs of serviceability failure ($C_f/C_1 = 1{,}000$) the optimum reliability levels reach the commonly recommended levels for the ultimate limit states. Obviously, an appropriate reliability level depends on the cost ratio C_f/C_1, which has to be assessed taking into account specific conditions of a particular structure.

10. Concluding Remarks

(1) Performance requirements are commonly specified by quantities of a random and vague nature.
(2) The damage function and damage density functions are basic fuzzy probabilistic tools for the description of random and vague serviceability requirements.
(3) The characteristic value or serviceability requirements may be defined as a fractile of the fuzzy probabilistic distribution, the probability $p = 0.05$ may not be the optimum value.
(4) The fuzzy probabilistic distribution of serviceability requirements may be used similarly as a classical distribution function to analyze reliability level and to optimize structural design.
(5) For water retaining structures the basic reinforcement area A (given by ultimate limit states) needs to be substantially increased to reach the limiting crack width.
(6) The optimum parameter $\omega_{opt} = A_{opt}/A_0$ increases with the cost ratio C_f/C_1 of the malfunctioning cost C_f to the cost per unit of the reinforcement area C_1 (for $C_f/C_1 = 1{,}000$, the optimum ω_{opt} can be about 4.0).

(7) The optimum reliability index can be expected within the broad interval from 0 to 3.5 depending on the cost ratio C_f / C_1.

(8) Further research should be focused also on the assessment of economic consequences of a serviceability failure when the crack width exceeds the limiting value.

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