THE VOLATILITY PREMIUM RISK: VALUATION AND FORECASTING[1]

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Abstract: *Empirical studies, such as Lamoureux and Lastrapes (1993), Guo (1998), Fouque et al. (2000) show that the market price of volatility risk is nonzero and time varying. This paper provides a theoretical investigation of the market price of volatility risk. We consider that the market price of volatility risk is a function of two variables: the price of underlying asset and its volatility. We suggest a closed-form solution for the price of volatility risk under the conditions of stochastic volatility and of correlation between the underlying asset price and its volatility. This formula involves in a direct way the unobservable market price of volatility risk. We prove that the correlation between underlying price and its volatility has no impact on the price of volatility risk. Finally, we present empirical results using the prices of CAC 40 index and of CAC 40 index call options from January 2006 to December 2007.*

Key words: *Premium risk; Volatility risk; Stochastic volatility; Option pricing; Risk forecasting*

1. Introduction

Option prices observed on a liquid option market contain rich information about the market's expectation of the future distribution of the underlying asset, and the risk premium for unhedgeable distribution risk.

If the market is complete, the derivative's risk can, theoretically, be perfectly hedged by the underlying asset and there is no volatility risk premium to be estimated. We consider that the volatility is stochastic and therefore not constant, the market is incomplete and consequently the price of volatility risk will be non-zero.

The assumption of zero correlation between volatility changes and aggregate consumption changes is often invoked to justify a zero risk price for volatility risk (Hull and White 1987). This assumption was challenged by Melino and Turnbull (1990), who found that a stochastic volatility model with no positive price of volatility risk explains observed prices better than the constant volatility models. Lamoureux and Lastrapes (1993) empirically showed that the market price of volatility risk is nonzero and time varying.

Moreover, a negative risk price parameter is consistent with the belief that volatility changes are negatively correlated with the aggregate consumption growth, and the investors prefer to pay a risk premium to hedge the volatility risk.

Various stochastic volatility option pricing models have been developed over the past few years. Many researchers, like Hull and White (1987), Stein and Stein (1991), Heston (1993), and Bakshi et al. (1997, 2000) concluded that the volatility of an asset's return could be itself a random variable describing a specific process. The model of Heston (1993) allows for the systematic volatility risk to be specified, whereas Hull and White (1987) and others have to assume a zero price of volatility risk in order to obtain a tractable option pricing model. Heston (1993) provided a closed-form solution for the price of a European-style option on an asset with mean-reverting square-root stochastic volatility. In the model of Heston, the dynamics of the underlying asset and the volatility are:

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t \quad (1)$$

$$d\sigma_t^2 = k\left(\theta - \sigma_t^2\right)dt + \sigma_v \sigma_t dW_t \quad (2)$$

where dB_t and dW_t are Wiener processes with instantaneous correlation ρ. The instantaneous variance $\sigma^2(t)$ follows a square-root process that was used by Cox, Ingersoll, and Ross (1985) to model the instantaneous interest rate process. The volatility diffusion process is a mean-reverting process with long-term mean parameter of θ, mean-reversion speed parameter of k, and the volatility of volatility parameter of σ_v.

Because there is no traded security than can be used to hedge the risk of volatility, it is difficult to form a risk-free portfolio. In this case, the option valuation is no longer preference free, and the market price of volatility risk needs to be determined. A formal treatment of the risk price requires an equilibrium model. In rational expectation equilibrium models of asset markets (Breeden 1979), a state variable's risk price is a function of its covariance with the representative investor's marginal utility of consumption, so $\gamma\left(S,\sigma^2,t\right) = \lambda COV\left(d\sigma^2, dC/C\right)$, where $C(t)$ is the consumption rate and λ is the coefficient of relative risk aversion of the representative investor. If percentage changes in consumption have constant covariance with the volatility changes, the price of the volatility risk can be represented as being proportional to volatility. Hence, Heston used the following condition: $\gamma\left(S,\sigma^2,t\right) = \gamma\sigma_t^2$. A negative price of volatility risk arises if this covariance is negative ($\gamma < 0$).

The market price of the volatility risk is an unobservable variable. We have only one piece of information about it: the volatility risk premium depends on the stock price and its volatility. In this paper we consider that the market price of volatility risk follows a diffusion process and depends on the underlying price and the volatility of underlying returns.

2. The Volatility Premium Risk Valuation

We consider an option pricing model with two state variables. Accordingly, the price of a European option depends on the price of the underlying asset and on the volatility, which describes the following stochastic process:

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t \quad (3)$$

$$d\sigma_t^2 = \varphi\sigma_t^2 dt + \sigma_v \sigma_t^2 dW_t \quad (4)$$

where the current values of S and σ^2 are known. The same stochastic definitions of the state variables are used by Hull and White (1987). The return drift, μ, is a parameter which depends on S (the price of the underlying asset), σ (the volatility), and time. The Brownian motions B_t and W_t are correlated ($dB_t dW_t = \rho dt$) and ρ is the correlation coefficient ($-1 < \rho < 1$). The volatility of the volatility (σ_v) and the drift of the variance (φ) are assumed to be constant, therefore implying that the non-anticipated variations of the variance are stationary.

Using the Girsanov's Theorem, under the risk-neutral probability equivalent with the real probability, the diffusion process described by the price of the underlying asset is given by:

$$dS_t = rS_t dt + \sigma_t S_t dB_t^* \quad (5)$$

where r is the risk-free return, B_t^* is the equivalent Brownian motion and $dB_t^* dW_t = \rho dt$.
We consider that the market price of volatility risk is a function of two state variables: the price of underlying asset and the volatility, $\gamma\left(S, \sigma^2, t\right)$. The price of volatility risk describes the following diffusion process:

$$d\gamma_t = \sigma_t dB_t^* + \sigma_v dW_t \quad (6)$$

Using the Itô's lemma, the dynamic of the market price of the volatility risk is given by:

$$d\gamma = \left[\frac{\partial\gamma}{\partial t} + \frac{\partial\gamma}{dS} rS + \frac{\partial\gamma}{\partial\sigma^2}\varphi\sigma^2 + \frac{1}{2}\frac{\partial^2\gamma}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2\gamma}{\partial\left(\sigma^2\right)^2}\sigma_v^2\sigma^4 + \frac{\partial^2\gamma}{\partial S\partial\sigma^2}\rho\sigma_v\sigma^3 S\right] + \frac{\partial\gamma}{\partial S}\sigma S dB + \frac{\partial\gamma}{\partial\sigma^2}\sigma_v\sigma^2 dW \quad (7)$$

Taking into account the stochastic process followed by the market price of the volatility risk, we obtain the following conditions:

$$\frac{\partial\gamma}{\partial t} + \frac{\partial\gamma}{dS} rS + \frac{\partial\gamma}{\partial\sigma^2}\varphi\sigma^2 + \frac{1}{2}\frac{\partial^2\gamma}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2\gamma}{\partial\left(\sigma^2\right)^2}\sigma_v^2\sigma^4 + \frac{\partial^2\gamma}{\partial S\partial\sigma^2}\rho\sigma_v\sigma^3 S = 0 \quad (8)$$

$$\frac{\partial\gamma}{\partial S}\sigma S = \sigma \quad (9)$$

$$\frac{\partial \gamma}{\partial \sigma^2}\sigma_v \sigma^2 = \sigma_v \tag{10}$$

This allows us to compute the partial derivatives: $\frac{\partial \gamma}{\partial S} = \frac{1}{S}$, $\frac{\partial \gamma}{\partial \sigma^2} = \frac{1}{\sigma^2}$, $\frac{\partial^2 \gamma}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial^2 \gamma}{\partial (\sigma^2)^2} = -\frac{1}{\sigma^4}$ and $\frac{\partial^2 \gamma}{\partial S \partial \sigma^2} = 0$. Finally, we obtain an ordinary differential equation by substituting the partial derivatives into the equation (8).

$$\frac{d\gamma}{dt} = -r - \varphi + \frac{1}{2}\sigma_v^2 + \frac{1}{2}\sigma^2(t) \tag{11}$$

with the initial condition: $\gamma(0) = 0$.

The solution of this equation gives the following formula for the valuation of the market price of volatility risk:

$$\gamma_t = \frac{1}{2}\left(\sigma_v^2 + \bar{V}\right)t - (r + \varphi)t \tag{12}$$

where $\bar{V} = \frac{1}{t}\int_0^t \sigma^2(u)\,du$, $\forall 0 < u < t$, represents the mean of historical variances. Moreover, the annualized market price of the volatility risk is:

$$\gamma_{at} = \frac{1}{2}\left(\sigma_v^2 + \bar{V}\right) - (r + \varphi) \tag{13}$$

Switching from t (the current date) to T (the future date), we can obtain the market price of volatility risk at the maturity of an option. The future price of volatility risk can be written as:

$$\gamma_T = \gamma_t + \frac{1}{2}\left(\sigma_v^2 + \tilde{\bar{V}}\right)\tau - (r + \varphi)\tau \tag{14}$$

where γ_t is known. The parameter τ is the time to maturity T, $\tau = T - t$. The mean of the futures variances between t and T is a random variable: $\tilde{\bar{V}} = \frac{1}{\tau}\int_t^T \sigma^2(u)\,du$, $\forall t < u < T$.

The obtained formula (14) can be used to forecast the market price of the volatility risk. Therefore,

$$E[\gamma_T] = \gamma_t + \frac{1}{2}\left(\sigma_v^2 + E\left[\tilde{\bar{V}}\right]\right)\tau - (r + \varphi)\tau \tag{15}$$

After some stochastic calculus, the expected value of the mean futures volatilities is given by:

$$E\left[\tilde{\bar{V}}\right] = \frac{e^{\varphi\tau} - 1}{\varphi\tau}\sigma^2 \tag{16}$$

The forecasting volatility premium risk is therefore given by:

$$E[\gamma_T] = \gamma_t + \frac{1}{2}\left(\sigma_v^2 + \frac{e^{\varphi\tau} - 1}{\varphi\tau}\sigma^2\right)\tau - (r + \varphi)\tau \tag{17}$$

We note that the volatility premium risk is inversely proportional to the instantaneous risk free return and no depends on the correlation between the asset price and its volatility.

3. Empirical Results

CAC 40 index series used in our studies were extracted from the Bourse de Paris database from July 1996 to December 1998. The database includes a time-stamped record of every trade occurred on the CAC 40 index. Our study focuses on the volatility premium risk of CAC 40 index traded in 2006 and 2007. On the sample, more than 35,000 quotations are reported.

The EURIBOR 3 months, which was the most liquid maturity on the French market, serves as a proxy for the risk free interest rate and is obtained from Datastream.

Option series used in these studies were extracted from the Bourse de Paris database from January 2006 to December 2007. Our study focuses on CAC 40 index call options (PXL contracts) traded at the MONEP in 2007 and 2008.

Options written on CAC 40 index are the most actively traded of the MONEP on the sample period. In 2006 and 2007, an average of 745,309 option contracts were traded monthly and 35,775 daily. Option premium reached EUR 750 million a month and EUR 35 million a day.

Over these years, both equity and index options accounted for a roughly similar proportion of lots traded, but equity options represented less than 25% of the total amount corresponding to the sum of premium.

A. Estimation of the volatility premium risk using historical volatility.

We estimate the market price of volatility risk of CAC 40 index from January 2006 to December 2007. Let x_i be the log-return on the index price defined as:

$$x_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \tag{18}$$

where S_i and S_{i-1} are respectively the index price at the end of the day i and of the day $i-1$, and x_i is the continuously compounded return (not annualized) of the index in the day i. Figure 1 shows the evolution of the continuously compounded return from July 2006 to December 2007.

We compute the historical volatility from January 2006 to December 2007 using the closing index prices from daily data over the recent 90 days. The historical volatility s_t is the standard deviation of the x_i's:

$$s_t = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2} \tag{19}$$

where $n = 90$ days, and $\overline{x}$ is the mean of index returns.

Assuming that time is measured in trading days and that there are 252 trading days per year, the volatility per annum at the date t, is:

$$\sigma_t = s_t \sqrt{252} \quad (20)$$

Figure 2 shows the evolution of historical volatility from October 2006 to December 2007.

We compute the mean of historical volatilities ($\overline{V}$) which occur in the last 90 days.

$$\overline{V} = \frac{1}{90}\sum_{i=1}^{90}\sigma_i^2 \quad (21)$$

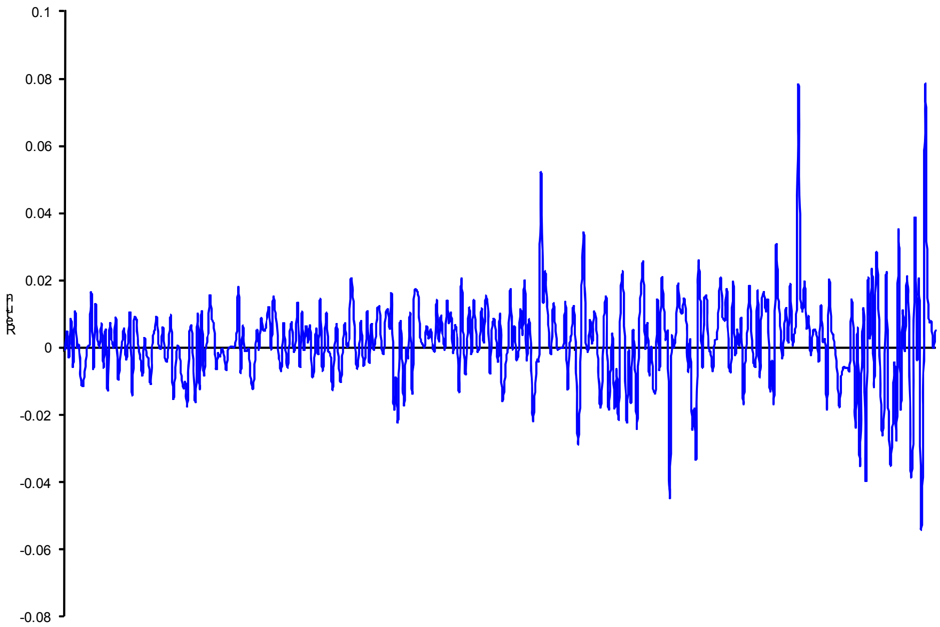

Figure 1. The Continuously Compounded Return

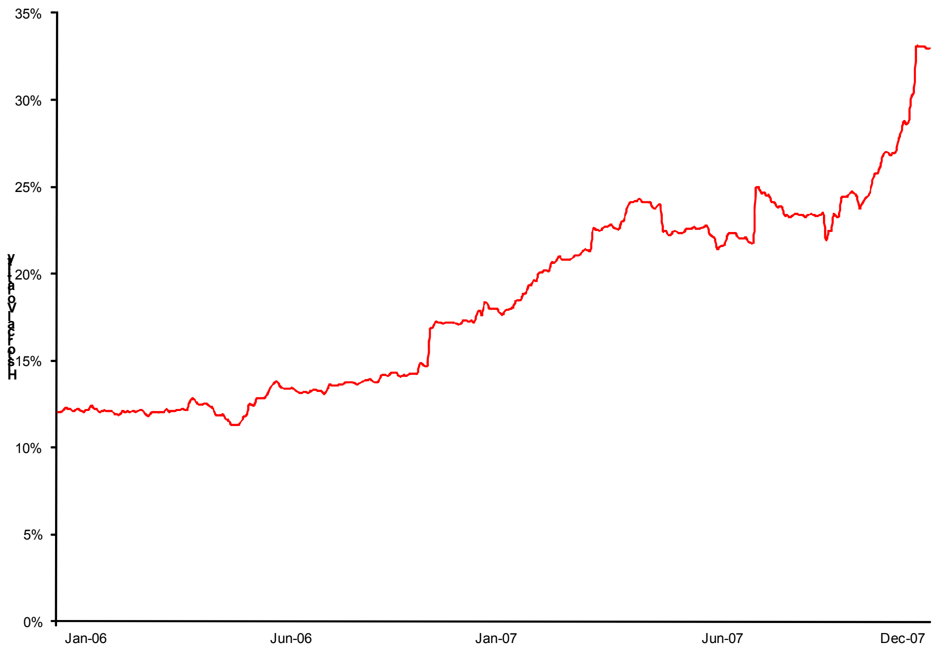

Figure 2. The Historical Volatility

It remains to estimate the volatility diffusion parameters, φ and σ_v, in order to apply the formula of the current price of volatility risk. We use the maximum likelihood method and the GMM estimation. The log-likelihood function is:

$$L = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln \sigma_v^2 h - \frac{1}{2}\frac{\left[\ln\frac{\sigma_t^2}{\sigma_{t-1}^2} - \left(\varphi - \frac{\sigma_v^2}{2}\right)h\right]^2}{\sigma_v^2 h} \quad (22)$$

In the case of GMM estimation we have the following conditions:

$$E_{t-1}\left[\varepsilon_t\right] = 0 \quad \text{and} \quad E_{t-1}\left[\varepsilon_t^2 - \sigma_v^2 h\right] = 0 \quad (23)$$

with

$$\varepsilon_t = \ln\frac{\sigma_t^2}{\sigma_{t-1}^2} - \left(\varphi - \frac{\sigma_v^2}{2}\right)h \quad (24)$$

We obtain the value of volatility diffusion parameters, φ and σ_v, at each date t between January 2006 and December 2007 using the time series of historical volatilities σ_t. Figure 3 shows the estimated volatility drift (φ) of CAC 40 index from January 2006 to December 2007.

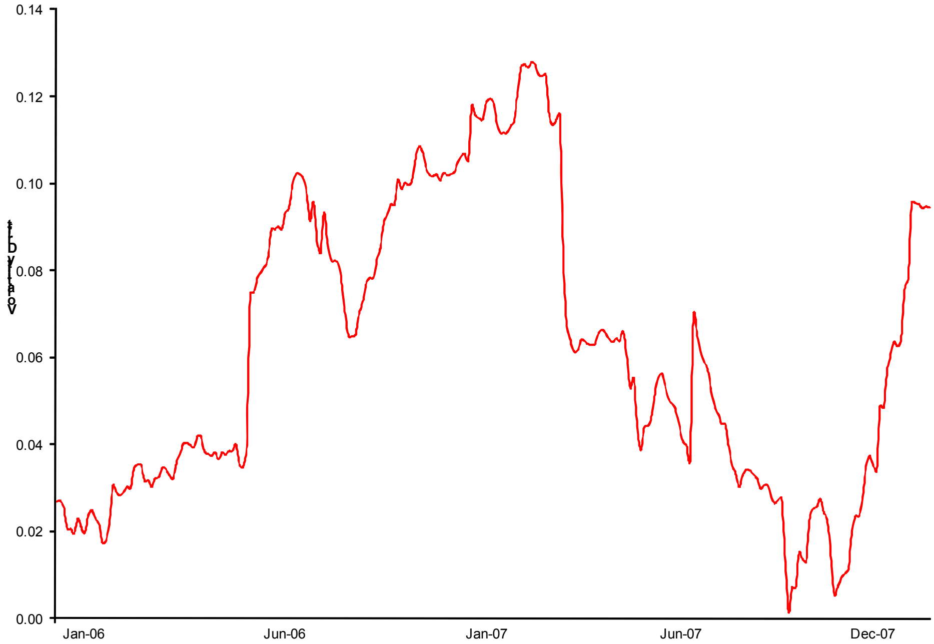

Figure 3. The Volatility Drift

Using the estimated parameters and applying the obtained formula, we compute the volatility premium risk. Its evolution is described in the Figure 4. We notice that the price of volatility risk is almost perfectly negatively correlated with the volatility drift. One unit of volatility premium decrease corresponds almost to one unit of volatility drift increase. If one buys calls with high volatility premium, the price paid for those options is generally higher. If a move up occurs, there is a chance that the underlying may not be able to move up fast enough to compensate for the decrease in volatility premium. We observe that the volatility premium may decrease as the price of the underlying increases.

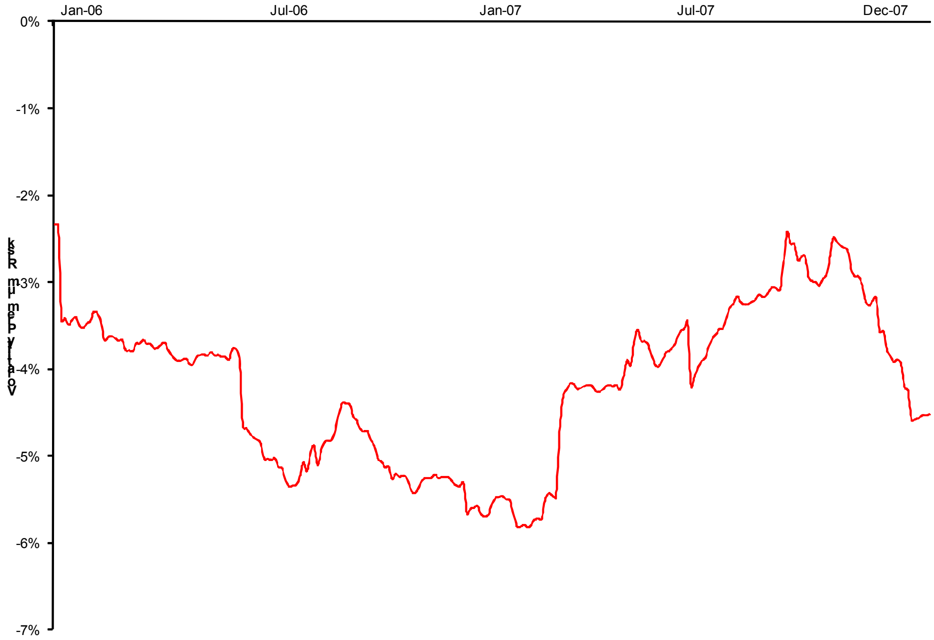

Figure 4. The Volatility Premium Risk Computed with Historical Volatility

B. Estimation of the volatility premium risk using implied volatility.

In order to compute the implied volatility we apply the Black-Scholes formula to the pricing of CAC 40 index call options traded at the MONEP. Reversing the Black-Scholes formula, we estimate the implied parameter (i.e. the implied volatility) for options negotiated on the MONEP.

The theoretical price of an option contract depends on four observable parameters: the strike price, the maturity, both specified in the contracts, the underlying price, and the risk free interest rate, which can be taken from public market data, as well as some non-observable parameters describing the risk-neutral density function. One parameter is required by the Black and Scholes model: the volatility. We use a non linear least squares procedure in both cases where n is the number of all call options j, $j=[1,...,n]$, available on a given day t, for a given maturity τ. The Black-Scholes call price is defined as $C_t^* = C_t(\hat{\sigma})$, where $\hat{\sigma}$ is the solution of the following minimization problem:

$$\hat{\sigma}_t = Arg\left\{ \underset{\hat{\sigma}}{Min}\left[\sum_{j=1}^{n} \left(C_j^{obs} - C_j^{BS}(\hat{\sigma}) \right)^2 \right] \right\} \quad (25)$$

Backing out implied parameter from all option prices on a day, we allow parameter to vary daily, which is inconsistent with the models assumption if parameter proved to be truly variable. Several empirical studies use this procedure (see, for instance, Bakshi et al., 1997 and Dumas et al., 1998).

Following the same technique described above, the volatility drift and the volatility of the volatility are computed using the implied volatility time series. Therefore, using the theoretical formula, we compute the market price of the volatility risk. Figure 5 shows the evolution of the volatility premium risk during the study period.

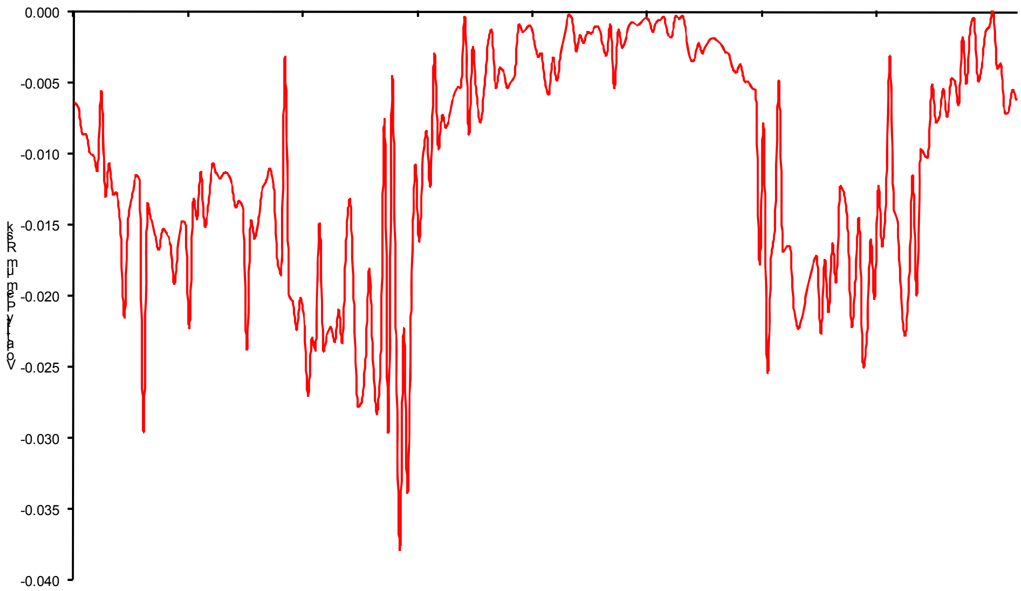

Figure 5. The Volatility Premium Risk Computed with Implied Volatility

We compute the forecasting price of volatility risk at each date t until date T using the expression (17).

The forecasted market price of volatility risk ($E\left[\gamma_T\right]$) of CAC 40 index is compared with the current market price of volatility risk (γ_t). The figure 6 describes the evolution of the current and of the forecasting prices of volatility risk from December, 22 2006 to Mars, 31 2007 while the figure 7 exhibits data from June, 24 2007 to September, 30 2007. The forecasted price of volatility risk and the current price of volatility risk are both negative.

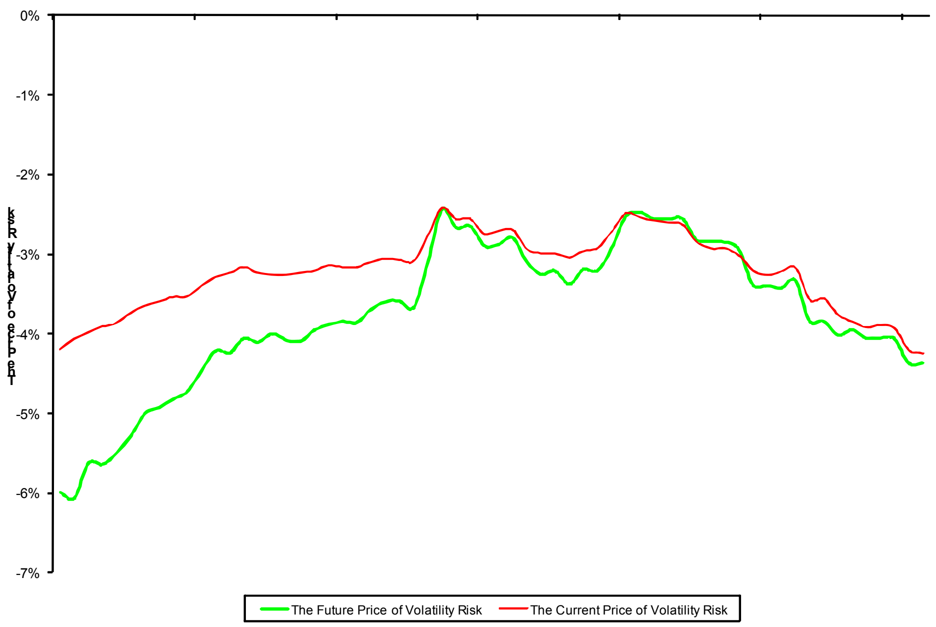

Figure 6. Forecasting versus Current Volatility Premium Risk

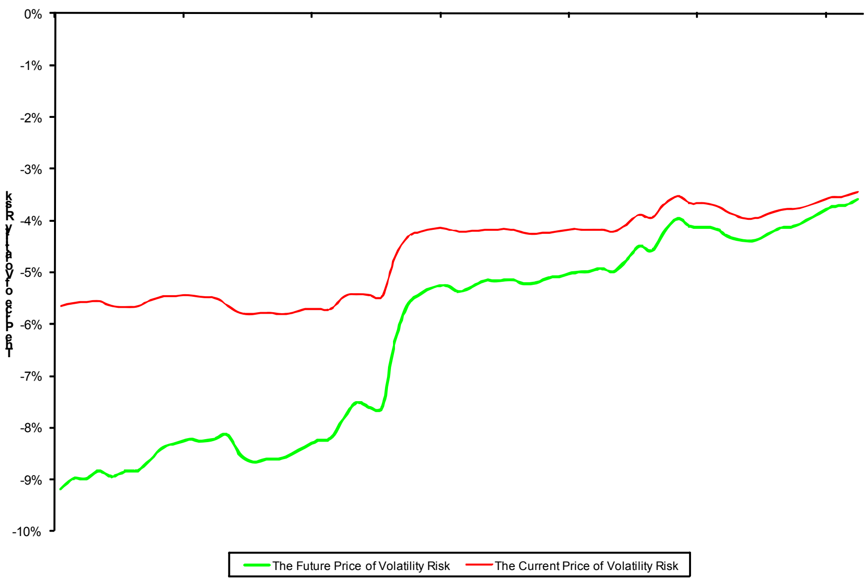

Figure 7. Forecasting versus Current Volatility Premium Risk

The forecasted price of volatility risk is lower than the current price of volatility risk. This explains the market's "crash-o-phobia". The forecasted volatility risk premium gives the possibility to quantify at each date the degree of "crash-o-phobia" of the market. In the first case, the forecasted price of volatility risk is -9.19% predictable for the 90th day in the future while the current price of volatility risk is -5.66%. In the second case, the forecasted price of volatility risk is -5.99% while the current price of volatility risk is -4.19%. The degree of market's "crash-o-phobia" was most important during the winter - spring period of 2007 than during the summer - autumn period of 2007.

4. Conclusions

We have shown that a model characterized by two state variables, the stock price and its volatility, allows us to obtain the volatility risk premium. By modeling the volatility and stock price processes, we obtain a formula for computing the price of volatility risk. This formula involves in a direct way the unobservable market price of volatility risk. We conclude that the volatility risk is determined by the volatility and by the diffusion parameters corrected by the risk-free interest rate. Hence, the historical data can be used to estimate the volatility risk premium.

The market price of volatility risk is negative. If one buys calls with high volatility premium, the price paid for those options is generally higher. If a move up occurs, there is a chance that the underlying may not be able to move up fast enough to compensate for the decrease in volatility premium. We may therefore notice a decrease in the volatility premium although the price of the stock goes up.

Lastly, the forecasted price of volatility risk can be used to measure the degree of market's "crash-o-phobia".

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