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SOME ASPECTS ON SOLVING A LINEAR FRACTIONAL TRANSPORTATION PROBLEM

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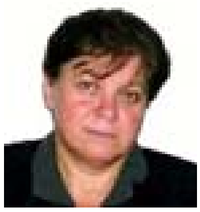

Abstract: *This paper presents the three-dimensional transportation problem, a double sum model in which the objective function is the ratio of two positive linear functions. This paper objective is to present how to obtain optimum with simplex method. To illustrate the procedure, a numerical example is given.*

Key words: *the three dimensional transportation problem; programming with fractional linear objective function; simplex method*

Problem Description

I am proposing now to solve the 3-dimensional transport problem – a double sum model - with the fractional linear objective function and linear constraints:

$$\min f(x) = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} m_{ijk} x_{ijk}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} n_{ijk} x_{ijk}} \qquad (1)$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijk} = a_i \qquad i = 1, m \qquad (2)$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p} x_{ijk} = b_j \qquad j = 1, n \qquad (3)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} = c_k \qquad k = 1, p \qquad (4)$$

$$x_{ijk} \geq 0 \qquad i = 1, m \qquad j = 1, n \qquad k = 1, p \qquad (5)$$

$$a_i, b_j, c_k > 0 \qquad (6)$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} c_k = T \qquad (7)$$

Requiring the following specifications:

m – the number of sources

n – the number of destinations

p – the number of means of conveyance

ai – the available quantity in each source i = 1,m

bj – the necessary quantity in each destination j = 1,n

ck – the quantity with must be transported by means of conveyance k = 1,p

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} m_{ijk} x_{ijk} \geq 0 \quad , \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} n_{ijk} x_{ijk} > 0$$

Matrix X = {x_{ijk} \ i = 1, m; j = 1, n; k = 1, p}, which satisfies constraints (2) (3) (4) (5), is called a transportation plan (feasible solution) and plan X is called optimum if it satisfies (1).

When the condition (7) is satisfied, the resulting formulation is called a balanced transportation problem. Relation (7) is the necessary and sufficient condition for the existence of the solution: the level of the matrix of the constraint system is m+n+p showing that a non-degenerated transportation plan of problem (1-7) contains at least m+n+p-2 non-null components;

The objective is to establish a transportation plan with minimum total expenses.

The function (1) is explicit quasi concave in S = { X / (2) (3) (4) (5) } i.e.:

If $x_1, x_2 \in S$, $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$, $\lambda \in (0,1)$ and $x_0 = \lambda x_1 + (1-\lambda)x_2$ then $\min[f(x_1), f(x_2)] < f(x_0)$.

For such function, local minimum is not necessarily a global minimum. Every differentiable [3] explicit quasi concave function is pseudo concave as well. An optimality criterion for local minimum is given in [1]

In this paper is made to generalize the results given by [2],[6]. This paper objective is to present how to obtain optimum with the help of the simplex method.

Solving the problem

The considerations concerning the three-dimensional problem are valid.

An initial feasible solution can be obtained by using the known methods from the three-dimensional transport problem [4][5].

We denote $I_x = \{(i,j,k) / x_{ijk} > 0, x_{ijk} \in X\}$

Due to (7) each nondegenerate solution will contain m+n+p–2 positive components.

We consider the dual variables (simplex multipliers):

u_i^1, u_i^2, $i = \overline{1,m}$, v_j^1, v_j^2, $j = \overline{1,n}$, w_k^1, w_k^2, $k = \overline{1,p}$

defined such that:

$$m_{ijk} = u_i^1 + v_j^1 + w_k^1 \tag{8}$$

$$n_{ijk} = u_i^2 + v_j^2 + w_k^2 \tag{9}$$

for (i, j, k) $\in I_x$ and

$$m'_{ijk} = m_{ijk} - (u_i^1 + v_j^1 + w_k^1) \qquad \forall (i,j,k) \tag{10}$$

$$n'_{ijk} = n_{ijk} - (u_i^2 + v_j^2 + w_k^2) \qquad \forall(i,j,k) \tag{11}$$

System (8) – (9) can be solved independently. So, system (8) – (9) has m+n+p-2 equations with m+n+p variables. We can arbitrarily set $u_1^1 = 0$, $v_1^1 = 0$ and solve for the other multipliers.

Having determined u_i^1, u_i^2, v_j^1, v_j^2, w_k^1, w_k^2 we shall use these values to determine m'_{ijk} and n'_{ijk} for the non-basic variables.

Let X* = $(x^*_{ijk})_{ijk}$ be a feasible solution of the problem (1) – (7).

To establish the optimal criterion we express f(x) in terms of the non-basic variables only.

$$\sum_i\sum_j\sum_k m_{ijk}x_{ijk} = \sum_i\sum_j\sum_k m_{ijk}x_{ijk} + \sum_i\left(a_i - \sum_j\sum_k x_{ijk}\right)u_i^1 + \sum_j\left(b_j - \sum_i\sum_k x_{ijk}\right)\cdot v_j^1 +$$

$$+\sum_k\left(c_k - \sum_i\sum_j x_{ijk}\right)\cdot w_k^1 =$$

$$= \sum_i\sum_j\sum_k\left(m_{ijk} - u_i^1 - v_j^1 - w_k^1\right)\cdot x_{ijk} + \sum_i a_i u_i^1 + \sum_j b_j v_j^1 + \sum_k c_k w_k^1 =$$

$$= \sum_i\sum_j\sum_k m'_{ijk}x_{ijk} + V_1$$

By means of a similar procedure we can also write :

$$\sum_i\sum_j\sum_k n_{ijk}x_{ijk} = \sum_i\sum_j\sum_k n'_{ijk}x_{ijk} + V_2$$

where Vs = $\sum_{i=1}^m a_i u_i^s + \sum_{j=1}^n b_j v_j^s + \sum_{k=1}^p c_k w_k^s$, s=1,2 (11)

Therefore the function f(x) becomes:

$$f(x) = \frac{\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^p m'_{ijk}x_{ijk} + V_1}{\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^p n'_{ijk}x_{ijk} + V_2}$$

For $\forall(i,j,k) \in I - I_x$ we have

$$\frac{\partial f}{\partial x_{ijk}} = \frac{m'_{ijk}\left(\sum_{I-I_x} n'_{ijk}x_{ijk} + V_2\right) - n'_{ijk}\left(\sum_{I-I_x} m'_{ijk}x_{ijk} + V_1\right)}{\left(\sum_{I-I_x} n'_{ijk}x_{ijk} + V_2\right)^2}$$

The partial derivates of the function f(x) evaluated at the point $x_{ijk} = x^*_{ijk}$ are:

$$\left.\frac{\partial f}{\partial x_{ijk}}\right|_{x_{ijk} = x^*_{ijk}} = \frac{m'_{ijk}V_2 - n'_{ijk}V_1}{(V_2)^2}$$

We note $\Delta_{ijk} = m'_{ijk}V_2 - n'_{ijk}V_1$

The dual variables u_i^1, u_i^2, v_j^1, v_j^2, w_k^1, w_k^2 $i=\overline{1,m}$ $j=\overline{1,n}$ $k=\overline{1,p}$ determined, it would be easy to calculate Δ_{ijk} for non-basic variables.

The solution X* can be improved if it exists at least a value $\Delta_{ijk} < 0$

Theorem A solution X* = $(x^*_{ijk})_{ijk}$ is a local optimum if $\Delta_{ijk} \geq 0$ for all non-basic variables.

If one of this values is not positive, we choose

$$\Delta_{i_0j_0k_0} = \min\{\Delta_{ijk} | \Delta_{ijk} < 0\}$$

and we improve the value of f(x) by introducing $x_{i_0j_0k_0}$ in the set of basic variables.

The variable which leaves the basis and the value of the basic variable in the basis can be determined as usual.

Example

Consider the problem with m=3, n=2, p=3, and
a_1=24, b_1=40, c_1=16
a_2=8, b_2=19, c_2=31
a_3=27, c_3=12
The matrices of costs: m_{ijk} / n_{ijk}

k=1

i \ j	1	2
1	15/8	9/3
2	19/20	15/14
3	14/10	24/20

k=2

i \ j	1	2
1	18/15	14/10
2	20/15	21/15
3	11/10	23/20

k=3

i \ j	1	2
1	7/3	11/5
2	17/10	14/10
3	8/5	13/6

An initial feasible solution obtained as in [4],[5] is X_0:

k=1

i \ j	1	2
1	16	
2		
3		

k=2

i \ j	1	2
1		8
2	8	
3	15	

k=3

i \ j	1	2
1		
2		
3	1	11

for which the value of objective function is $f_0 = \frac{828}{549} \approx 1{,}5$

Optimality verification : we determine the quantities u_i^1, v_j^1, w_k^1, u_i^2, v_j^2, w_k^2, $i=\overline{1,3}$ $j=\overline{1,2}$ $k=\overline{1,3}$

from systems:

$u_1^1 + v_1^1 + w_1^1 = 15$ $u_1^2 + v_1^2 + w_1^2 = 8$

$u_1^1+v_2^1+w_2^1=14$ $\quad$ $u_1^2+v_2^2+w_2^2=10$

$u_2^1+v_1^1+w_2^1=20$ $\quad$ $u_2^2+v_1^2+w_2^2=15$

$u_3^1+v_1^1+w_2^1=11$ $\quad$ $u_3^2+v_1^2+w_2^2=10$

$u_3^1+v_1^1+w_3^1=8$ $\quad$ $u_3^2+v_1^2+w_3^2=5$

$u_3^1+v_2^1+w_3^1=13$ $\quad$ $u_3^2+v_2^2+w_3^2=6$

We obtain

$u_1^1=0$ $\quad v_1^1=0$ $\quad w_1^1=15$ $\quad u_1^2=0$ $\quad v_1^2=0$ $\quad w_1^2=8$

$u_2^1=11$ $\quad v_2^1=5$ $\quad w_2^1=9$ $\quad u_2^2=6$ $\quad v_2^2=1$ $\quad w_2^2=9$

$u_3^1=2$ $\quad w_3^1=6$ $\quad u_3^2=1$ $\quad w_3^2=4$

The matrices: m'_{ijk} / n'_{ijk}

k=1

i \ j	1	2
1	0/0	-11/-6
2	-7/6	-16/-1
3	-3/-1	2/10

k=2

i \ j	1	2
1	9/6	0/0
2	0/0	-4/-1
3	0/0	7/9

k=3

i \ j	1	2
1	1/-1	0/0
2	0/0	-8/-1
3	0/0	0/0

For which $V_1=828$

$V_2=549$, $V_1\approx 1{,}5V_2$

And matrix $\Delta_{ijk}=m'_{ijk}\cdot V_2-n'_{ijk}\cdot V_1\approx V_2(m'_{ijk}-1{,}5\cdot n'_{ijk})=V_2\cdot\Delta'_{ijk}$

($\Delta_{ijk}=0$ for basic components).

Matrix Δ'_{ijk} :

k=1

i \ j	1	2
1	0	-2
2	-16	-14,5
3	-4,5	-13

k=2

i \ j	1	2
1	0	0
2	0	-2,5
3	0	-6,5

k=3

i \ j	1	2
1	2,5	0
2	0	-6,5
3	0	0

The solution is not optimum because there are components Δ_{ijk} < 0.

We improved solution:

Input criterion : $\Delta_{211}=-16=\min\{\Delta_{ijk}\,|\,\Delta_{ijk}<0\}$

Output criterion: as in [4],[5]: for basic components :

$z_{111}=1$ $\quad z_{212}=1$ $\quad z_{313}=-1$

$z_{122}=-1$ $\quad z_{322}=0$ $\quad z_{323}=1$

$\theta=\min(x_{111},x_{212},x_{323})=\min(16,8,11)=8$ $\qquad x_{212}=\theta=8$

Solution actualization for basic components:

$$x'_{ijk} = x_{ijk} - \theta \cdot z_{ijk}$$

$$x_{212} = 0$$

$$x_{211} = 8$$

The new solution X_1 is:

k=1

i \ j	1	2
1	8	
2	8	
3		

k=2

i \ j	1	2
1		16
2		
3	15	

k=3

i \ j	1	2
1		
2		
3	9	3

for which the value of objective function is $f_1 = \frac{772}{597} \approx 1,3$

Resume from optimality verification

Bibliography

1. Aggarwal, S. P. **Indefinite quadratic fractional programming,** Cahiers du Centre d'Etudes de Recherche Operationelle, 1973
2. Corban, A. **Non Linear three-dimensional programming,** Rev. Roum. Math. Pures et Appl., 1975
3. Dantzig, G. B., Wolfe, F. **The Decomposition Algorithm for Linear Programs,** Econometrica, vol. 29, no. 4, 1961
4. Mangasarian, O. L. **Pseudo Convex Functions,** J. Siam Control, nr.3, 1965
5. Moanta, D. **Three dimensional transport problems,** Denbridge Press, New York, 2006
6. Moanta, D. **Teorie si practica in probleme de transport,** Cartea Universitara, Bucharest, 2006
7. Stancu Minasian, I. M. **A Three-Dimensional Transportation Problem with a Special Structured Objective Function,** Bulletin Math., Tome 18, no. 3-4, 1974

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Recently published books:

1. Moanta D: Three dimensional transport problems, Denbridge Press, New York, 2006
2. Moanta D: Teorie si practica in probleme de transport, Cartea Universitara, Bucharest, 2006