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A QUASI GARIMA DISTRIBUTION

Rama SHANKER
Department of Statistics,
Eritrea Institute of Technology, Asmara, Eritrea

E-mail: shankerrama2009@gmail.com

Kamlesh Kumar SHUKLA
Department of Statistics,
Eritrea Institute of Technology, Asmara, Eritrea

E-mail: kkshukla22@gmail.com

Simon SIUM
Department of Statistics,
Eritrea Institute of Technology, Asmara, Eritrea

Daniel WELDAY
Department of Statistics,
Eritrea Institute of Technology, Asmara, Eritrea

Ravi SHANKER
Department of Mathematics,
G.L.A. College, N.P University, Daltonganj, Jharkhand, India

E-mail: ravi.shanker74@gmail.com

Abstract
In the present paper, a two-parameter quasi Garima distribution (QGD) which includes one parameter exponential distribution and Garima distribution introduced by Shanker (2016 c) as special cases, has been proposed. Its statistical and mathematical properties including moments and moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy measure and stress-strength reliability have also been discussed. The method of moments and the method of maximum likelihood estimation have been discussed for estimating the parameters of QGD. Finally, the goodness of fit of the QGD has been discussed with a real lifetime dataset and the fit is quite satisfactory over one parameter and two-parameter lifetime distributions.

Keywords: *Garima distribution; Moments; Reliability Properties; Stochastic ordering; Mean deviations; Stress-strength reliability; Estimation of parameters; goodness of fit*

1. Introduction

Shanker (2016 c) has introduced a one parameter lifetime distribution named Garima distribution for modeling lifetime data from behavioral science having probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(x;\theta)=\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x} \quad ;x>0,\ \theta>0. \tag{1.1}$$

$$F_1(x,\theta)=1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x} \ ;x>0,\theta>0 \tag{1.2}$$

Shanker (2016 c) has shown that it gives better fit than one parameter exponential distribution, Lindley distribution introduced by Lindley (1958) and Shanker, Akash, Aradhana and Sujatha distributions introduced by Shanker (2015 a, 2015 b, 2016 a, 2016 b). This distribution is a convex combination of exponential (θ) and gamma $(2,\theta)$ distributions with their mixing proportion $\frac{\theta+1}{\theta+2}$.

The first four moments about origin of Garima distribution obtained by Shanker (2016 c) are given as

$$\mu_1' = \frac{\theta+3}{\theta(\theta+2)}, \quad \mu_2' = \frac{2(\theta+4)}{\theta^2(\theta+2)}, \quad \mu_3' = \frac{6(\theta+5)}{\theta^3(\theta+2)}, \quad \mu_4' = \frac{24(\theta+6)}{\theta^4(\theta+2)}$$

The central moments of Garima distribution obtained by Shanker (2016 c) are

$$\mu_2 = \frac{\theta^2+6\theta+7}{\theta^2(\theta+2)^2}$$

$$\mu_3 = \frac{2(\theta^3+9\theta^2+21\theta+15)}{\theta^3(\theta+2)^3}$$

$$\mu_4 = \frac{3(3\theta^4+36\theta^3+134\theta^2+204\theta+111)}{\theta^4(\theta+2)^4}$$

Shanker (2016 c) studied its important properties including coefficient of variation, skewness, kurtosis, Index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, and stress-strength reliability. Shanker (2016 c) has also discussed the estimation of parameter using both the method of moments and the method of maximum likelihood estimation and the application of the distribution to model behavioral science data. The discrete Poisson – Garima distribution, a Poisson mixture of Garima distribution has also been studied by Shanker (2017).

In this paper, a two - parameter quasi Garima distribution (QGD), of which one parameter exponential distribution and Garima distribution introduced by Shanker (2016 c) are particular cases, has been proposed. Its raw moments and central moments have been obtained and coefficients of variation, skewness, kurtosis and index of dispersion have been discussed. Some of its important mathematical and statistical properties including hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy measure and stress-strength reliability have also been discussed. The estimation of the parameters has been discussed using both the method of moments and the maximum likelihood estimation. The goodness of fit of QGD has been illustrated with a real lifetime dataset and the fit has been compared with well known one parameter and two-parameter lifetime distributions.

2. A quasi Garima distribution

A two - parameter quasi Garima distribution (QGD) having parameters θ and α is defined by its pdf

$$f_2\left(x;\theta,\alpha\right)=\frac{\theta^2}{\theta^2+\theta+\alpha}\left(1+\theta+\alpha x\right)e^{-\theta x}\ ;x>0,\ \theta>0,\alpha>0. \qquad (2.1)$$

It can be easily verified that (2.1) reduces to the exponential distribution and Garima distribution at $\alpha=0$ and $\alpha=\theta$ respectively. It can be easily verified that QGD is a convex combination of exponential (θ) and gamma $(2,\theta)$ distributions. We have

$$f_2\left(x;\theta,\alpha\right)=p\,g_1\left(x;\theta\right)+(1-p)\,g_2\left(x;2,\theta\right) \qquad (2.2)$$

where

$$p=\frac{(\theta+1)\theta}{\theta^2+\theta+\alpha},$$

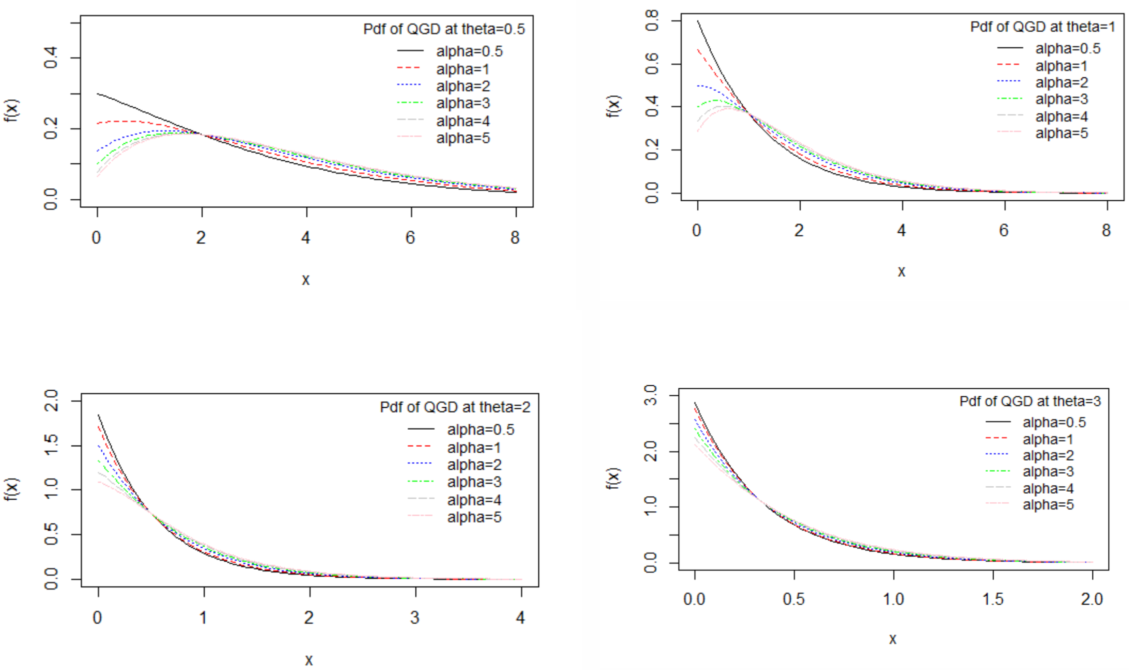
$$g_1\left(x;\theta\right)=\theta\, e^{-\theta x};\ \ x>0,\ \theta>0$$

$$g_2\left(x;\theta\right)=\frac{\theta^2}{\Gamma\left(2\right)}e^{-\theta x}x^{2-1};\ \ x>0,\ \theta>0\,.$$

The corresponding cdf of QGD (2.1) can be obtained as

$$F_2\left(x;\theta,\alpha\right)=1-\left[1+\frac{\alpha\,\theta x}{\theta^2+\theta+\alpha}\right]e^{-\theta x};\ \ x>0,\theta>0,\alpha>0 \qquad (2.3)$$

The nature and behavior of the pdf and the cdf of QGD for varying values of the parameters θ and α have been explained graphically and presented in figures 1 and 2, respectively. From fig. 1, it is obvious that when θ is fixed and α is changing, there is a slight difference in the shapes of the pdf of QGD. Further, when α is fixed and θ is changing, there is a remarkable difference in the shapes of the pdf of QGD. This means that the parameter θ is playing a dominant role in the shape of the pdf of QGD. The same fact can be observed from the shapes of the cdf of QGD in fig. 2.



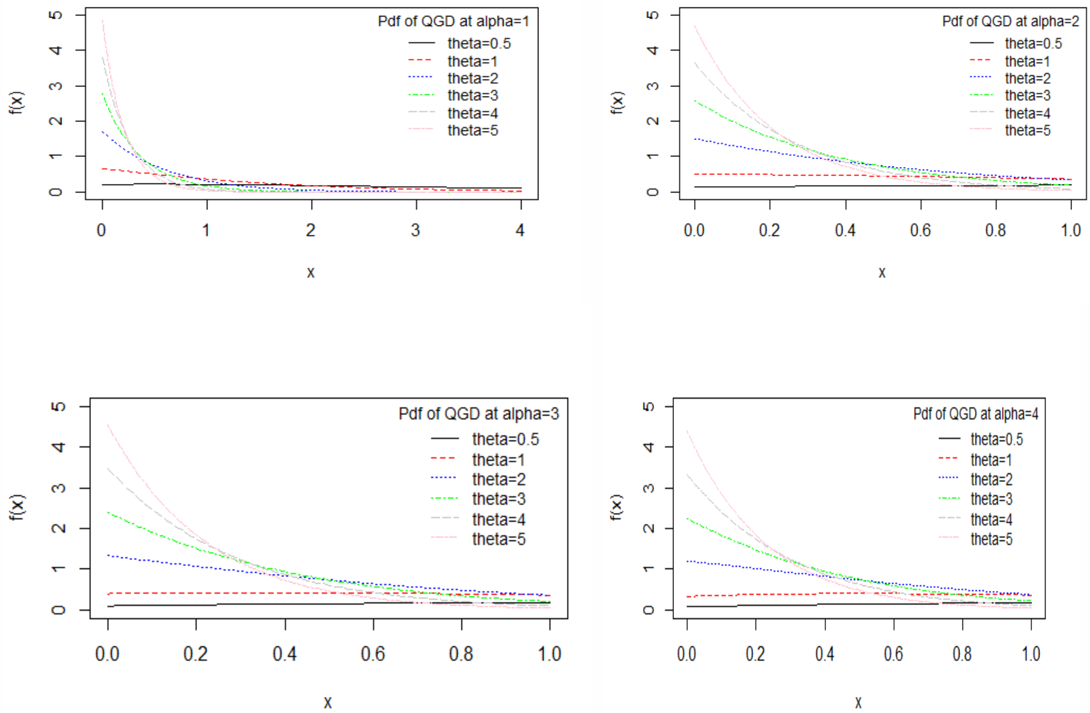
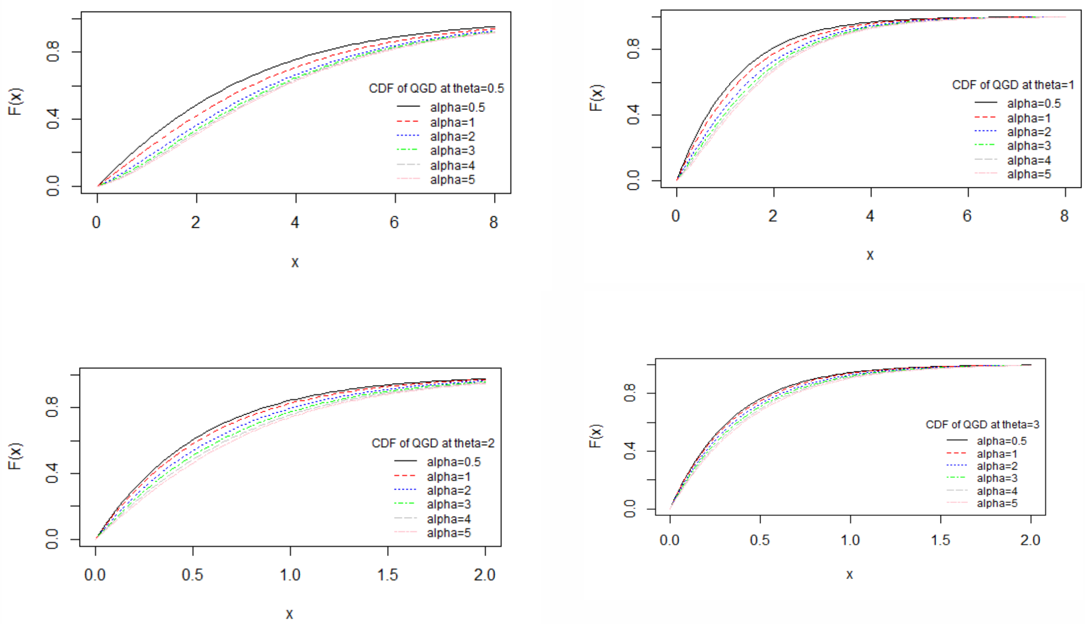


Figure 1. Graphs of the pdf of QGD for varying values of parameters θ and α



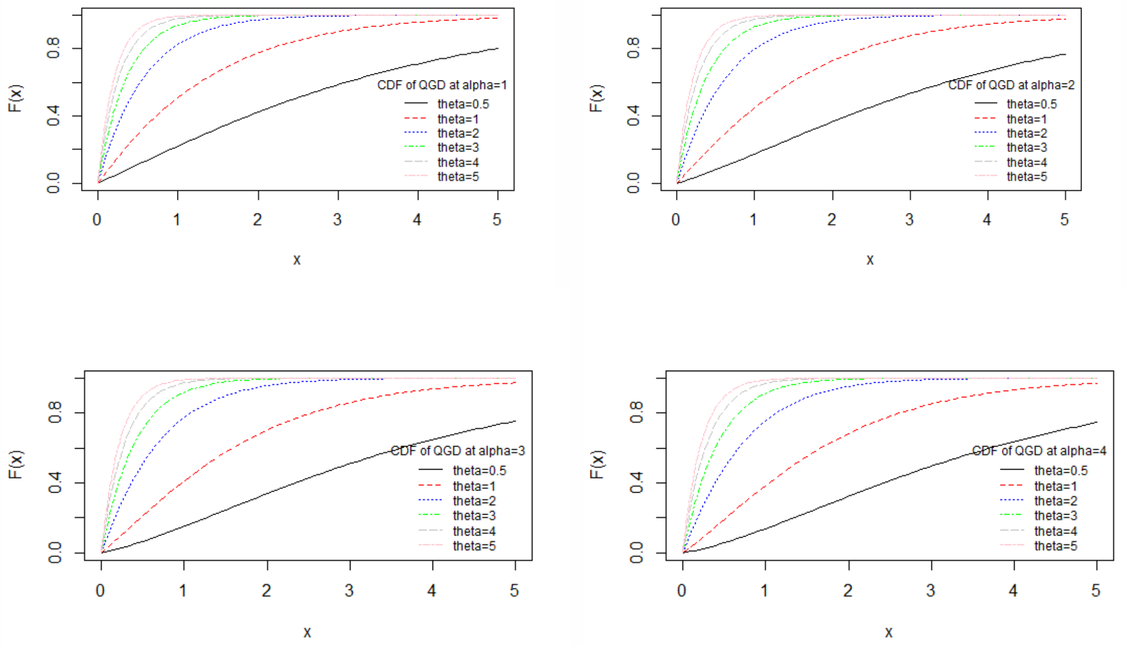


Figure 2. Graphs of the cdf of QGD for varying values of parameters θ and α

3. Statistical constants

Using the convex combination representation (2.2), the rth moment about origin of QGD (2.1) can be obtained as

$$\mu_r' = \frac{r!\left\{\theta^2+\theta+(r+1)\alpha\right\}}{\theta^r\left(\theta^2+\theta+\alpha\right)} \quad ; r=1,2,3,... \tag{3.1}$$

Thus, the first four moments about origin of QGD are given by

$$\mu_1' = \frac{\theta^2+\theta+2\alpha}{\theta\left(\theta^2+\theta+\alpha\right)}, \qquad \mu_2' = \frac{2\left(\theta^2+\theta+3\alpha\right)}{\theta^2\left(\theta^2+\theta+\alpha\right)}$$

$$\mu_3' = \frac{6\left(\theta^2+\theta+4\alpha\right)}{\theta^3\left(\theta^2+\theta+\alpha\right)}, \qquad \mu_4' = \frac{24\left(\theta^2+\theta+5\alpha\right)}{\theta^4\left(\theta^2+\theta+\alpha\right)}$$

Using relationship between central moments and moments about origin, the central moments of QGD are thus obtained as

$$\mu_2 = \frac{\theta^4+2\theta^3+(4\alpha+1)\theta^2+4\theta\alpha+2\alpha^2}{\theta^2\left(\theta^2+\theta+\alpha\right)^2}$$

$$\mu_3 = \frac{2\left\{\theta^6+3\theta^5+3(2\alpha+1)\theta^4+(12\alpha+1)\theta^3+6\alpha(\alpha+1)\theta^2+6\theta\alpha^2+2\alpha^3\right\}}{\theta^3\left(\theta^2+\theta+\alpha\right)^3}$$

$$\mu_4 = \frac{3\begin{Bmatrix} 3\theta^8 + 12\theta^7 + 6(4\alpha+3)\theta^6 + 12(6\alpha+1)\theta^5 + \left(44\alpha^2 + 72\alpha + 3\right)\theta^4 \\ +8(11\alpha+3)\theta^3\alpha + 4(8\alpha+11)\theta^2\alpha^2 + 32\theta\alpha^3 + 8\alpha^4 \end{Bmatrix}}{\theta^4\left(\theta^2+\theta+\alpha\right)^4}.$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of QGD are obtained as

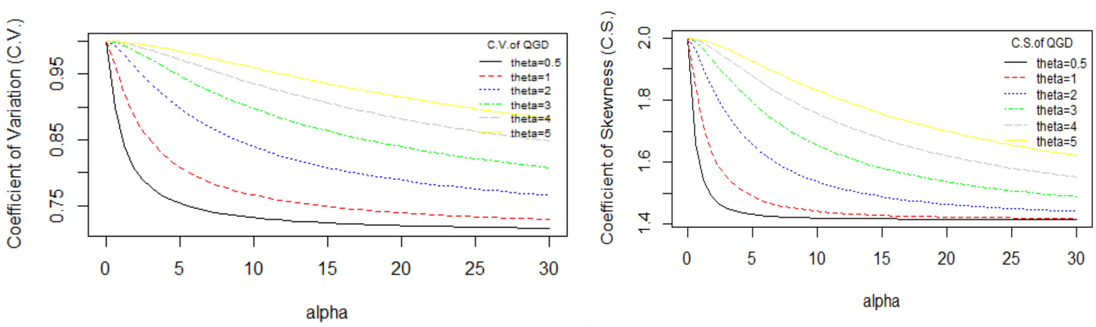
$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4 + 2\theta^3 + (4\alpha+1)\theta^2 + 4\theta\alpha + 2\alpha^2}}{\theta^2+\theta+2\alpha}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left\{\theta^6 + 3\theta^5 + 3(2\alpha+1)\theta^4 + (12\alpha+1)\theta^3 + 6\alpha(\alpha+1)\theta^2 + 6\theta\alpha^2 + 2\alpha^3\right\}}{\left\{\theta^4 + 2\theta^3 + (4\alpha+1)\theta^2 + 4\theta\alpha + 2\alpha^2\right\}^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\begin{Bmatrix} 3\theta^8 + 12\theta^7 + 6(4\alpha+3)\theta^6 + 12(6\alpha+1)\theta^5 + \left(44\alpha^2 + 72\alpha + 3\right)\theta^4 \\ +8(11\alpha+3)\theta^3\alpha + 4(8\alpha+11)\theta^2\alpha^2 + 32\theta\alpha^3 + 8\alpha^4 \end{Bmatrix}}{\left\{\theta^4 + 2\theta^3 + (4\alpha+1)\theta^2 + 4\theta\alpha + 2\alpha^2\right\}^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4 + 2\theta^3 + (4\alpha+1)\theta^2 + 4\theta\alpha + 2\alpha^2}{\theta\left(\theta^2+\theta+\alpha\right)\left(\theta^2+\theta+2\alpha\right)}.$$

Graphs of C.V, $\sqrt{\beta_1}$, β_2 and γ of QGD for varying values of the parameters θ and α have been presented in figure 3.The C.V is monotonically decreasing for increasing values of the parameters θ and α but for increasing value of the parameter θ, the C.V shifts upward. The nature of coefficient of skewness (C.S) is also similar to the nature of C.V. The coefficient of kurtosis (C.K) is also monotonically decreasing for increasing values of the parameters θ and α but for $\theta = 0.5$ and $\alpha \geq 2$, the C.K become constant. QGD is over-dispersed $\left(m < s^2\right)$, equi-dispersed $\left(m = s^2\right)$ and under-dispersed $\left(m > s^2\right)$ for $\left(q < 1, a\ ^3\ 0\right)$, $\left(q = 1, a\ ^3\ 0\right)$ and $\left(q > 1, a\ ^3\ 0\right)$ respectively, which is obvious from the graphs of index of dispersion (I.D) in figure 3.



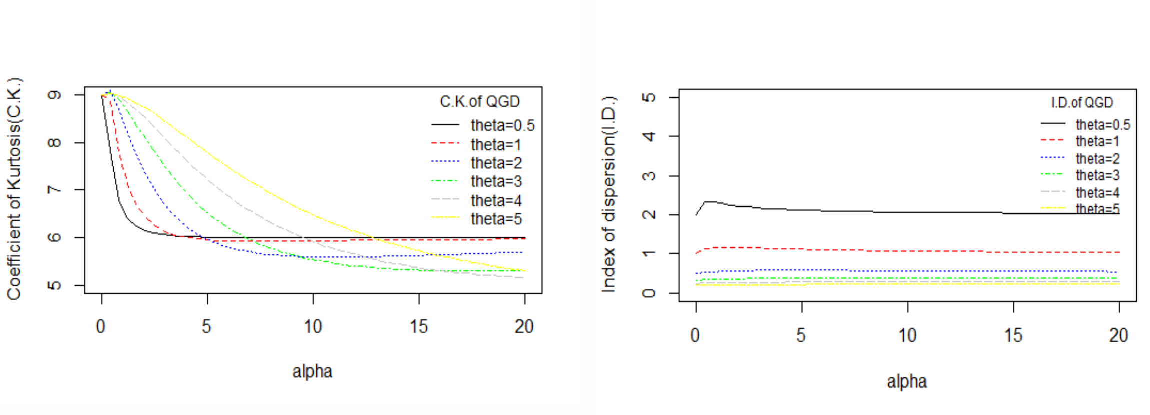


Figure 3. Graphs of C.V, $\sqrt{\beta_1}$, β_2 and γ of QGD for varying values of parameters θ and α

4. Reliability properties

Suppose X is a continuous random variable with pdf $f(x)$ and cdf $F(x)$. The hazard rate function (also known as the failure rate function) $h(x)$ and the mean residual life function $m(x)$ of X are respectively defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P(X<x+\Delta x|X>x)}{\Delta x}=\frac{f(x)}{1-F(x)} \tag{4.1}$$

and $m(x)=E\left[X-x\,|\,X>x\right]=\dfrac{1}{1-F(x)}\displaystyle\int_x^\infty\left[1-F(t)\right]\,dt$ (4.2)

Thus corresponding $h(x)$ and $m(x)$ of QGD are thus obtained as

$$h(x)=\frac{f_2(x;\theta,\alpha)}{1-F_2(x;\theta,\alpha)}=\frac{\theta^2(\alpha x+\theta+1)}{\alpha\theta x+\theta^2+\theta+\alpha} \tag{4.3}$$

and $m(x)=\dfrac{1}{\left(\alpha\theta x+\theta^2+\theta+\alpha\right)e^{-\theta x}}\displaystyle\int_x^\infty\left(\alpha\theta t+\theta^2+\theta+\alpha\right)e^{-\theta t}dt$

$$=\frac{\alpha\theta x+\theta^2+\theta+2\alpha}{\theta\left(\alpha\theta x+\theta^2+\theta+\alpha\right)} \tag{4.4}$$

It can be easily verified that $h(0)=\dfrac{\theta^2(\theta+1)}{\theta^2+\theta+\alpha}=f(0)$ and $m(0)=\dfrac{\theta^2+\theta+2\alpha}{\theta\left(\theta^2+\theta+\alpha\right)}=\mu_1'$.

The nature and behavior of $h(x)$ and $m(x)$ of QGD for varying values of parameters θ and α have been shown graphically in figures 4 and 5. For fixed values of α

and increasing values of $\boldsymbol{q}$, the $h(x)$ is shifting upward with very minor decrease /increase , whereas for fixed $\boldsymbol{q}$ and increasing $\boldsymbol{a}$, $h(x)$ is monotonically increasing.

The graphs of $m(x)$ are monotonically increasing for increasing values of $\boldsymbol{q}$ and $\boldsymbol{a}$, which is obvious from fig. 4.

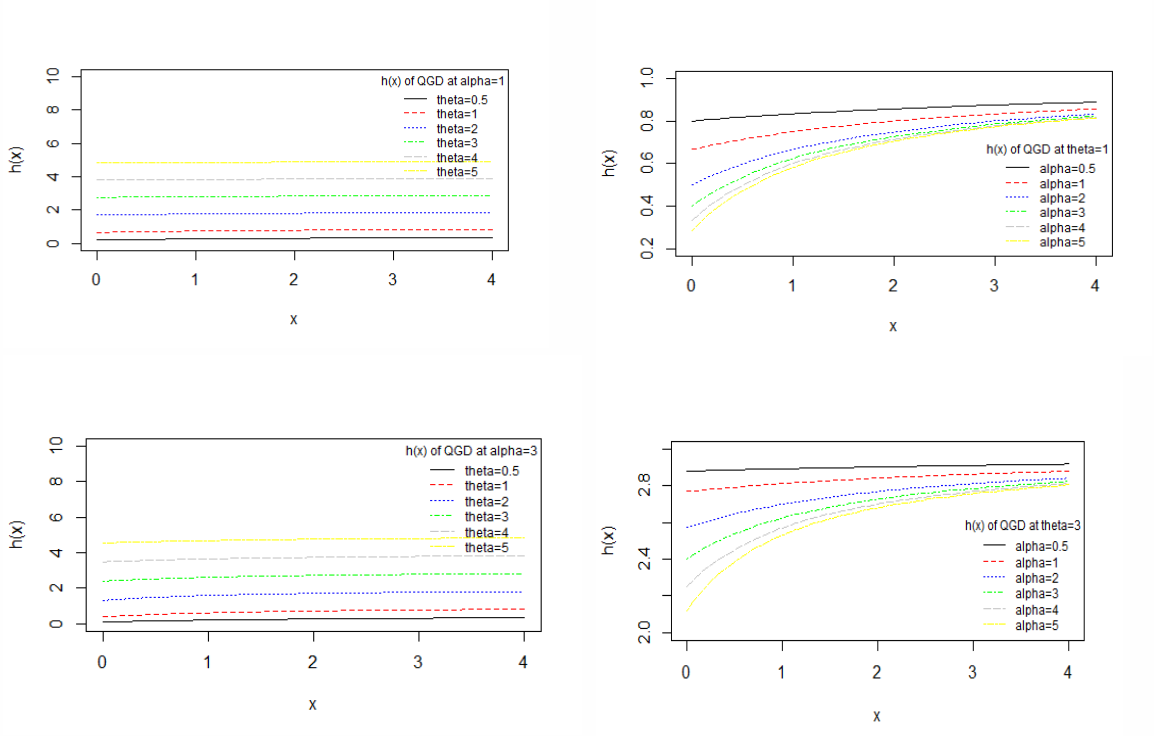


Figure 3. Graphs of $h(x)$ of QGD for varying values of parameters θ and α

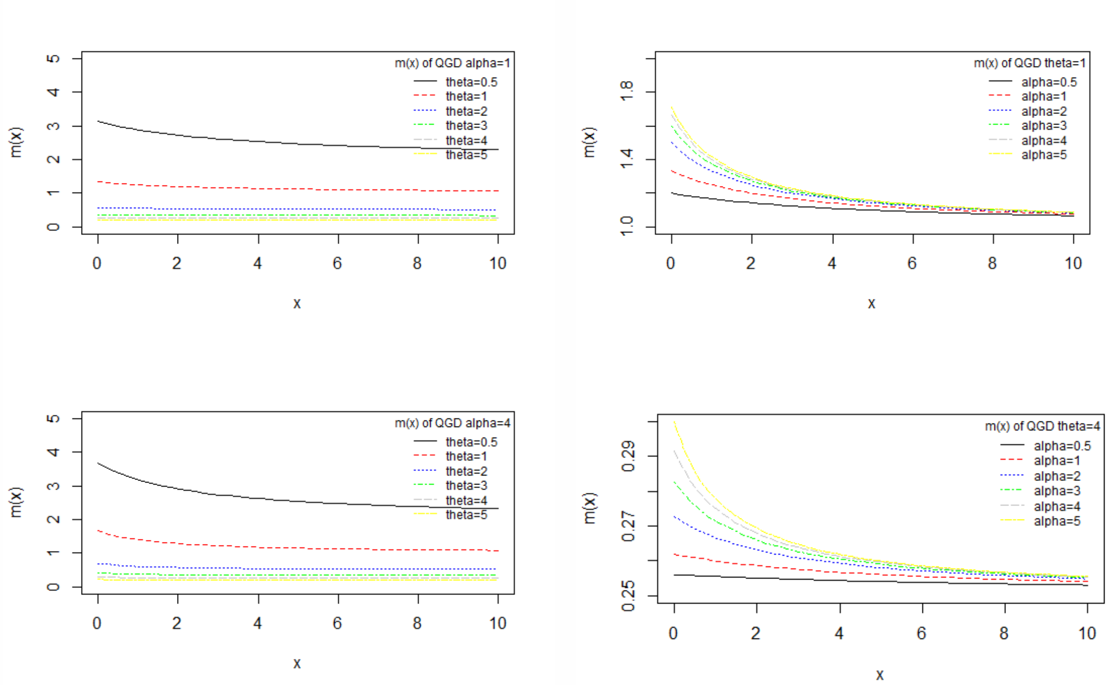


Figure 4. Graphs of $m(x)$ of QGD for varying values of parameters θ and α

5. Stochastic ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $(X \le_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

The QGD is ordered with respect to the strongest 'likelihood ratio ordering' as shown in the following theorem:

Theorem: Let $X \sim$ QGD(θ_1, α_1) and $Y \sim$ QGD(θ_2, α_2). If $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ (or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^2\left(\theta_2^2+\theta_2+\alpha_2\right)}{\theta_2^2\left(\theta_1^2+\theta_1+\alpha_1\right)}\left(\frac{1+\theta_1+\alpha_1 x}{1+\theta_2+\alpha_2 x}\right)e^{-(\theta_1-\theta_2)x} \quad ; x>0$$

Now

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left[\frac{\theta_1^2\left(\theta_2^2+\theta_2+\alpha_2\right)}{\theta_2^2\left(\theta_1^2+\theta_1+\alpha_1\right)}\right] + \ln\left(\frac{1+\theta_1+\alpha_1 x}{1+\theta_2+\alpha_2 x}\right) - (\theta_1-\theta_2)x.$$

This gives

$$\frac{d}{dx}\left\{\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)}\right\} = \frac{(\alpha_1-\alpha_2)+(\alpha_1\theta_2-\alpha_2\theta_1)}{(1+\theta_1+\alpha_1 x)(1+\theta_2+\alpha_2 x)} - (\theta_1-\theta_2).$$

Thus if $\alpha_1=\alpha_2$ and $\theta_1>\theta_2$ or $\theta_1=\theta_2$ and $\alpha_1<\alpha_2$, $\dfrac{d}{dx}\ln\dfrac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)}<0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

6. Mean deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined by

$\delta_1(X)=\int_0^\infty |x-\mu|\, f(x)\,dx$ and $\delta_2(X)=\int_0^\infty |x-M|\, f(x)\,dx$, respectively, where $\mu=E(X)$ and $M=\text{Median }(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following simplified relationships

$$\delta_1(X)=\int_0^\mu (\mu-x)f(x)\,dx+\int_\mu^\infty (x-\mu)f(x)\,dx$$

$$=\mu F(\mu)-\int_0^\mu x\, f(x)\,dx-\mu\left[1-F(\mu)\right]+\int_\mu^\infty x\, f(x)\,dx$$

$$=2\mu F(\mu)-2\mu+2\int_\mu^\infty x\, f(x)\,dx$$

$$=2\mu F(\mu)-2\int_0^\mu x\, f(x)\,dx \tag{6.1}$$

and

$$\delta_2(X)=\int_0^M (M-x)f(x)\,dx+\int_M^\infty (x-M)f(x)\,dx$$

$$=M\,F(M)-\int_0^M x\, f(x)\,dx-M\left[1-F(M)\right]+\int_M^\infty x\, f(x)\,dx$$

$$=-\mu+2\int_M^\infty x\, f(x)\,dx$$

$$=\mu-2\int_0^M x\, f(x)\,dx \tag{6.2}$$

Using pdf of QGD (2.1) and expression for the mean of QGD, we get

$$\int_0^\mu x\, f_2(x;\theta,\alpha)\,dx=\mu-\frac{\left\{\alpha\,\theta^2\mu^2+\theta\left(\theta^2+\theta+2\alpha\right)\mu+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta\mu}}{\theta\left(\theta^2+\theta+\alpha\right)} \tag{6.3}$$

$$\int_0^M x\, f_2(x;\theta,\alpha)\,dx=\mu-\frac{\left\{\alpha\,\theta^2M^2+\theta\left(\theta^2+\theta+2\alpha\right)M+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta M}}{\theta\left(\theta^2+\theta+\alpha\right)} \tag{6.4}$$

Using expressions from (6.1), (6.2), (6.3), and (6.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of QGD are finally obtained as

$$\delta_1(X)=\frac{2\left\{\alpha\,\theta\,\mu+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta\mu}}{\theta\left(\theta^2+\theta+\alpha\right)} \tag{6.5}$$

$$\delta_2(X)=\frac{2\left\{\alpha\theta^2M^2+\theta\left(\theta^2+\theta+2\alpha\right)M+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta M}}{\theta\left(\theta^2+\theta+\alpha\right)}-\mu \tag{6.6}$$

7. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves (Bonferroni (1930)) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx=\frac{1}{p\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{7.1}$$

and $$L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx=\frac{1}{\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{7.2}$$

respectively or equivalently

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{7.3}$$

and $$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{7.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{7.5}$$

and $$G=1-2\int_0^1 L(p)\,dp \tag{7.6}$$

respectively.

Using pdf of QGD (2.1), we get

$$\int_q^\infty x\,f_2(x;\theta,\alpha)\,dx=\frac{\left\{\alpha\theta^2q^2+\theta\left(\theta^2+\theta+2\alpha\right)q+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta q}}{\theta\left(\theta^2+\theta+\alpha\right)} \tag{7.7}$$

Now using equation (7.7) in (7.1) and (7.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\alpha\theta^2q^2+\theta\left(\theta^2+\theta+2\alpha\right)q+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta q}}{\theta^2+\theta+2\alpha}\right] \tag{7.8}$$

and

$$L(p)=1-\frac{\left\{\alpha\theta^2q^2+\theta\left(\theta^2+\theta+2\alpha\right)q+\left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta q}}{\theta^2+\theta+2\alpha} \tag{7.9}$$

Now using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of QGD are thus obtained as

$$B = 1 - \frac{\left\{\alpha\theta^2 q^2 + \theta\left(\theta^2+\theta+2\alpha\right)q + \left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta q}}{\theta^2+\theta+2\alpha} \tag{7.10}$$

$$G = \frac{2\left\{\alpha\theta^2 q^2 + \theta\left(\theta^2+\theta+2\alpha\right)q + \left(\theta^2+\theta+2\alpha\right)\right\}e^{-\theta q}}{\theta^2+\theta+2\alpha} - 1 \tag{7.11}$$

8. Order statistics and Renyi entropy measure

8.1. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from QGD (2.1). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The pdf and the cdf of the kth order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1-F(y)\}^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l F^{k+l-1}(y) f(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\{1-F(y)\}^{n-j}$$

$$= \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l F^{j+l}(y),$$

respectively, for $k = 1, 2, 3, ..., n$.

Thus, the pdf and the cdf of kth order statistic of QGD are thus obtained as

$$f_Y(y) = \frac{n!\theta^2(1+\theta+\alpha x)e^{-\theta x}}{\left(\theta^2+\theta+\alpha\right)(k-1)!(n-k)!} \times \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \times \left[1 - \frac{(\theta^2+\theta+\alpha)+\alpha\theta x}{\theta^2+\theta+\alpha} e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y) = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l \left[1 - \frac{(\theta^2+\theta+\alpha)+\alpha\theta x}{\theta^2+\theta+\alpha}\right]^{j+l}$$

8.2. Renyi Entropy Measure

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy (1961). If X is a continuous random variable having pdf $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}$$

where $\gamma > 0$ and $\gamma \neq 1$.

Thus, the Renyi entropy of QGD can be obtained as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{2\gamma}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}(1+\theta+\alpha x)^{\gamma} e^{-\theta\gamma x}dx\right]$$

$$= \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{2\gamma}(1+\theta)^{\gamma}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}\left(1+\frac{\alpha x}{1+\theta}\right)^{\gamma} e^{-\theta\gamma x}dx\right]$$

$$= \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{2\gamma}(1+\theta)^{\gamma}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\left(\frac{\alpha x}{1+\theta}\right)^{j} e^{-\theta\gamma x}dx\right]$$

$$= \frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{j}\theta^{2\gamma}(1+\theta)^{\gamma-j}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}\int_0^{\infty} e^{-\theta\gamma x}x^{j+1-1}dx\right]$$

$$= \frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{j}\theta^{2\gamma}(1+\theta)^{\gamma-j}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}\frac{\Gamma(j+1)}{(\theta\gamma)^{j+1}}\right]$$

$$= \frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma}\theta^{2\gamma-j-1}\left(1+\theta\right)^{\gamma-j}}{\left(\theta^2+\theta+\alpha\right)^{\gamma}}\frac{\Gamma(j+1)}{(\gamma)^{j+1}}\right]$$

9. Stress-strength reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till

$X > Y$. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having QGD (2.1) with parameter (θ_1, α_1) and (θ_2, α_2) respectively. Then the stress-strength reliability R of QGD (2.1) can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_X(x)\,dx$$

$$= \int_0^\infty f_2(x;\theta_1,\alpha_1)\, F_2(x;\theta_2,\alpha_2)\,dx$$

$$= 1 - \frac{\theta_1^2 \begin{bmatrix} (\theta_1+\theta_2)^2(\theta_1+1)\left(\theta_2^2+\theta_2+\alpha_2\right)+2(\alpha_1\alpha_2\theta_2) \\ +(\theta_1+\theta_2)\left\{\alpha_2\theta_2(\theta_1+1)+\alpha_1\left(\theta_2^2+\theta_2+\alpha_2\right)\right\} \end{bmatrix}}{\left(\theta_1^2+\theta_1+\alpha_1\right)\left(\theta_2^2+\theta_2+\alpha_2\right)(\theta_1+\theta_2)^3}.$$

It can be easily verified that at $\alpha_1 = \theta_1$ and $\alpha_2 = \theta_2$, the above expression reduces to R of Garima distribution.

10. Estimation of parameters

10.1. Method of Moment Estimates (MOME)

Since QGD have two parameters to be estimated, the first two moments about origin are required to get method of moment estimates. Equating the sample mean to the corresponding population mean, we get

$$\bar{x} = \frac{\theta^2+\theta+2\alpha}{\theta\left(\theta^2+\theta+\alpha\right)} = \frac{1}{\theta} + \frac{\alpha}{\theta^2+\theta+\alpha}$$

This gives

$$\theta^2+\theta+\alpha = \frac{\alpha}{\theta\bar{x}-1} \qquad (9.1.1)$$

Again equating the second sample moment to corresponding second population moment, we get

$$m_2' = \frac{2\left(\theta^2+\theta+3\alpha\right)}{\theta^2\left(\theta^2+\theta+\alpha\right)} = \frac{2}{\theta^2} + \frac{4\alpha}{\theta^2\left(\theta^2+\theta+\alpha\right)}$$

This gives

$$\theta^2+\theta+\alpha = \frac{4\alpha}{m_2'\theta^2-2} \qquad (9.1.2)$$

Using equations (9.1.1) and (9.1.2), we get a quadratic equation in θ as

$$m_2'\theta^2 - 4\bar{x}\theta + 2 = 0.$$

This gives the MOME estimate $\tilde{\theta}$ of θ as

$$\tilde{\theta} = \frac{2\bar{x} + \sqrt{4\bar{x}^2 - 2m_2'}}{m_2'} \quad ; m_2' < 2\bar{x}^2$$

Substituting the value of $\tilde{\theta}$ in equation (9.1.1) , the MOME $\tilde{\alpha}$ of α is given by

$$\tilde{\alpha} = \frac{\theta(\theta+1)(\theta\bar{x}-1)}{2-\theta\bar{x}}$$

10.2. Maximum Likelihood Estimates (MLE)

Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n from QGD (2.1)). The likelihood function, L of QGD is given by

$$L = \left(\frac{\theta^2}{\theta^2+\theta+\alpha}\right)^n \prod_{i=1}^{n}(1+\theta+\alpha x_i)e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta^2}{\theta^2+\theta+\alpha}\right) + \sum_{i=1}^{n}\ln(1+\theta+\alpha x_i) - n\theta\bar{x}$$

The maximum likelihood estimates (MLE) $(\hat{\theta}, \hat{\alpha})$ of (θ, α) are then the solutions of the following non-linear equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - \frac{n(2\theta+1)}{\theta^2+\theta+\alpha} - n\bar{x} + \sum_{i=1}^{n}\frac{1}{1+\theta+\alpha x_i} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{-n}{\theta^2+\theta+\alpha} + \sum_{i=1}^{n}\frac{x_i}{1+\theta+\alpha x_i} = 0$$

where $\bar{x}$ is the sample mean.

These two natural log likelihood equations do not seem to be solved directly because they are not in closed forms. However, the Fisher's scoring method can be applied to solve these equations. For, we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n(2\theta^2+2\theta-2\alpha+1)\alpha^2}{(\theta^2+\theta+\alpha)^2} - \sum_{i=1}^{n}\frac{1}{(1+\theta+\alpha x_i)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = \frac{n(2\theta+1)}{(\theta^2+\theta+\alpha)^2} - \sum_{i=1}^{n}\frac{x_i}{(1+\theta+\alpha x_i)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{n}{(\theta^2+\theta+\alpha)^2} - \sum_{i=1}^{n}\frac{x_i^2}{(1+\theta+\alpha x_i)^2}$$

The solution of following equations gives MLE's $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of QGD

$$\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \theta^2} & \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L}{\partial \theta} \\ \frac{\partial \ln L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α as given by the MOME of QGD. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

11. Data analysis

In this section, the goodness of fit of QGD has been discussed with a real lifetime dataset from engineering and the fit has been compared with one parameter and two-parameter lifetime distributions. The following dataset represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, Lawless (2003)

1.4 5.1 6.3 10.8 12.1 18.5 19.7 22.2 23.0 30.6 37.3 46.3
53.9 59.8 66.2

In order to compare the considered distributions, values of $-2\ln L$, AIC(Akaike Information Criterion) and K-S Statistic (Kolmogorov-Smirnov Statistic) and p-value for the dataset have been computed and presented in table 2. The AIC and K-S Statistic are defined as follow:

$AIC = -2\ln L + 2k$ and $K\text{-}S = \sup_x \left| F_n(x) - F_0(x) \right|$, where k =number of parameters, n= sample size, $F_n(x)$ is the empirical distribution function and $F_0(x)$ is the theoretical cumulative distribution function.. The best distribution corresponds to the lower values of $-2\ln L$, AIC, K-S statistic and higher p-value.

The pdf and the cdf of the fitted distributions have been given in table 1. Recall that the quasi Shanker distribution (QSD) has been introduced by Shanker & Shukla (2017) and the exponentiated exponential distribution (EED) has been introduced by Gupta & Kundu (1999). The Lindley distribution has been introduced by Lindley (1958) and its detailed study has been done by Ghitany *et al* (2008).

Table 1. The pdf and the cdf of the fitted distributions

Models	p.d.f.	c.d.f.
QSD	$f(x;\theta,\alpha) = \left(\frac{\theta^3}{\theta^3+\theta+2\alpha}\right) \times \left(\theta + x + \alpha x^2\right) e^{-\theta x}$	$F(x;\theta,\alpha) = 1 - \left[1 + \frac{\left\{\begin{matrix}\alpha\theta^2 x^2 \\ +\theta x(\theta+2\alpha)\end{matrix}\right\}}{\theta^3+\theta+2\alpha}\right] e^{-\theta x}$
Weibull	$f(x;\theta,\alpha) = \theta\alpha x^{\alpha-1} e^{-\theta x^{\alpha}}$	$F(x;\theta,\alpha) = 1 - e^{-\theta x^{\alpha}}$

Gamma	$f(x;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}$	$F(x;\theta,\alpha)=1-\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)}$
Lognormal	$f(x;\theta,\alpha)=\frac{1}{\sqrt{2\pi\alpha x}}e^{-\frac{1}{2}\left(\frac{\log x-\theta}{\alpha}\right)^2}$	$F(x;\theta,\alpha)=\phi\left(\frac{\log x-\theta}{\alpha}\right)$
EED	$f(x;\theta,\alpha)=\alpha\theta\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x}$	$F(x;\theta,\alpha)=\left(1-e^{-\theta x}\right)^{\alpha}$
Lindley	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$F(x;\theta)=1-\left[\frac{\theta+1+\theta x}{\theta+1}\right]e^{-\theta x}$
Exponential	$f(x;\theta)=\theta e^{-\theta x}$	$F(x;\theta)=1-e^{-\theta x}$

Table 2. MLE's, $-2\ln L$, Standard Error (S.E), AIC, K-S Statistics and p-value of the fitted distributions of dataset.

Distributions	ML Estimates	S.E	$-2\ln L$	AIC	K-S	p-value
QGD	$\hat{\theta}=0.06225$	0.01721	128.21	132.21	0.095	0.997
	$\hat{\alpha}=0.16577$	0.32075				
QSD	$\hat{\theta}=0.07389$	0.04131	129.37	133.37	0.121	0.961
	$\hat{\alpha}=0.00147$	0.04401				
Gamma	$\hat{\theta}=0.05236$	0.02067	128.37	132.37	0.102	0.992
	$\hat{\alpha}=1.44219$	0.47771				
Weibull	$\hat{\theta}=0.01190$	0.01124	128.04	132.04	0.098	0.995
	$\hat{\alpha}=1.30586$	0.24925				
Lognormal	$\hat{\theta}=2.93059$	0.26472	131.23	135.23	0.161	0.951
	$\hat{\alpha}=1.02527$	0.18718				
EED	$\hat{\theta}=0.04529$	0.01372	128.47	132.47	0.108	0.986
	$\hat{\alpha}=1.44347$	0.51301				
Garima	$\hat{\theta}=0.05462$	0.01227	128.52	130.52	0.123	0.954
Lindley	$\hat{\theta}=0.07022$	0.01283	128.81	130.81	0.110	0.983
Exponential	$\hat{\theta}=0.03631$	0.00937	129.47	131.47	0.156	0.807

It can be easily seen from table 2 that the QGD gives better fit than one parameter exponential, Lindley and Garima distributions and two-parameter QSD, Gamma, Weibull lognormal and EED and hence it can be considered as an important distribution for modeling lifetime dataset over these distributions.

12. Concluding remarks

A two-parameter quasi Garima distribution (QGD), of which one parameter exponential distribution and Garima distribution introduced by Shanker (2016 c) are a particular cases, has been suggested and investigated. Its mathematical properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy measure and stress-strength reliability have been discussed.

For estimating its parameters the method of moments and the method of maximum likelihood estimation have been discussed. Finally, a numerical example of real lifetime dataset has been presented to test the goodness of fit of QGD over one parameter exponential, Lindley and Garima distributions and two-parameter QSD, Gamma, Weibull lognormal and EED and the fit by QGD has been found to be quite satisfactory. Therefore, QGD can be recommended as an important two-parameter lifetime distribution for modeling lifetime data over these distributions.

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