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THE POISSON-WEIGHTED AKASH DISTRIBUTION AND ITS APPLICATIONS

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Abstract

A Poisson-Weighted Akash distribution which includes Poisson-Akash distribution has been proposed. Its moments and moments based statistical constants have been derived and studied. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Finally, applications of the proposed distribution have been explained through two count datasets and the goodness of fit has been compared with other discrete distributions.

Keywords: *Weighted Akash distribution; Poisson- Akash distribution; Compounding; Moments; Skewness; Kurtosis; Maximum likelihood estimation; Applications*

1. Introduction

Shanker (2017) introduced the discrete Poisson- Akash distribution (PAD) to model count data defined by its probability mass function (pmf)

$$P_1(x;\theta)=\frac{\theta^3}{\theta^2+2}\cdot\frac{x^2+3x+\left(\theta^2+2\theta+3\right)}{\left(\theta+1\right)^{x+3}}\;;x=0,1,2,...,\theta>0 \tag{1.1}$$

Moments and moments based measures, statistical properties; estimation of parameter using both the method of moment and the method of maximum likelihood and applications of PAD has been discussed by Shanker (2017). The PAD arises from the Poisson distribution when its parameter λ follows Akash distribution introduced by Shanker (2015) defined by its probability density function (pdf)

$$f_1(\lambda,\theta)=\frac{\theta^3}{\theta^2+2}\left(1+\lambda^2\right)e^{-\theta\lambda}\;;\lambda>0,\theta>0 \tag{1.2}$$

The pdf (1.2) is a convex combination of exponential (θ) and gamma $(3,\theta)$ distributions. Shanker (2015) discussed statistical properties including moments based coefficients, hazard rate function, mean residual life function, mean deviations, stochastic ordering, Renyi entropy measure, order statistics, Bonferroni and Lorenz curves, stress- strength reliability, along with estimation of parameter and applications of Akash distribution to model lifetime data from biomedical science and engineering.

The first four moments about origin and the variance of PAD (1.1) obtained by Shanker (2017) are given by

$$\mu_1' = \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}$$

$$\mu_2' = \frac{\theta^3+2\theta^2+6\theta+24}{\theta^2\left(\theta^2+2\right)}$$

$$\mu_3' = \frac{\theta^4+6\theta^3+12\theta^2+72\theta+120}{\theta^3\left(\theta^2+2\right)}$$

$$\mu_4' = \frac{\theta^5+14\theta^4+42\theta^3+192\theta^2+720\theta+720}{\theta^4\left(\theta^2+2\right)}$$

$$\mu_2 = \sigma^2 = \frac{\theta^5+\theta^4+8\theta^3+16\theta^2+12\theta+12}{\theta^2\left(\theta^2+2\right)^2}.$$

Sankaran (1970) proposed the Poisson-Lindley distribution (PLD) to model count data defined by its pmf

$$P_2\left(x;\theta\right)=\frac{\theta^2\left(x+\theta+2\right)}{\left(\theta+1\right)^{x+3}} \quad ;x=0,1,2,...,\theta>0 \qquad (1.3)$$

Shanker and Hagos (2015) proposed a simple method of finding moments of PLD and discussed the applications of PLD to model count data from biological sciences. The PLD arises from the Poisson distribution when its parameter λ follows Lindley (1958) distribution defined by its probability density function (pdf)

$$f_2\left(\lambda;\theta\right)=\frac{\theta^2}{\theta+1}\left(1+\lambda\right)e^{-\theta\lambda} \; ;\lambda>0,\theta>0 \qquad (1.4)$$

It can be easily verified that the pdf (1.4) is a convex combination of exponential (θ) and gamma $(2,\theta)$ distributions. Ghitany *et al* (2008) discussed statistical properties including moments based coefficients, hazard rate function, mean residual life function, mean deviations, stochastic ordering, Renyi entropy measure, order statistics, Bonferroni and Lorenz curves, stress- strength reliability, along with estimation of parameter and application of Lindley distribution to model waiting time data in a bank. Shanker *et al* (2015) have detailed study on modeling of various lifetime data from engineering and biomedical sciences using exponential and Lindley distribution and observed that there are many lifetime data where exponential distribution gives much better fit than Lindley distribution.

Ghitany *et al* (2011) introduced a two-parameter weighted Lindley distribution (WLD) having parameters θ and α and defined by its pdf

$$f_3(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.5}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-y}\,y^{\alpha-1}\,dy;\ \alpha>0$ is the complete gamma function. Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications for modeling survival time data has been discussed by Ghitany *et al* (2011). The corresponding cumulative distribution function (cdf) of WLD (1.5) is given by

$$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^\alpha e^{-\theta x}}{(\theta+\alpha)\Gamma(\alpha)}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.6}$$

where

$$\Gamma(\alpha,z)=\int_z^\infty e^{-y}\,y^{\alpha-1}\,dy;\alpha>0,\ z\geq 0 \tag{1.7}$$

is the upper incomplete gamma function. It can be easily shown that at $\alpha=1$, WLD (1.5) reduces to Lindley (1958) distribution (1.4). Shanker *et al* (2016) discussed various moments based properties including coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of weighted Lindley distribution and its applications to model lifetime data from biomedical sciences and engineering. Shanker *et al* (2017) have proposed a three-parameter weighted Lindley distribution (TPWLD) which includes a two-parameter weighted Lindley distribution and one parameter Lindley distribution as particular cases and discussed its various structural properties, estimation of parameters and applications for modeling lifetime data from engineering and biomedical sciences.

Assuming that the parameter λ of the Poisson distribution follows WLD (1.5), El-Monsef and Sohsah (2014) proposed Poisson- weighted Lindley distribution (P-WLD) defined by its pmf

$$P_3(x;\theta,\alpha)=\frac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}};x=0,1,2,...,\theta>0,\alpha>0 \tag{1.8}$$

It can be easily verified that PLD (1.3) is a particular case of P-WLD for $\alpha=1$.

Shanker and Shukla (2016) proposed a two-parameter weighted Akash distribution (WAD) having parameters θ and α and defined by its pdf

$$f_4(x;\theta,\alpha)=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x^2)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.9}$$

Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications for modeling survival time data has been discussed by Shanker and Shukla (2016). It can be easily shown that at $\alpha=1$, WAD (1.9) reduces to Akash distribution (1.2).

The main purpose of this paper is to introduce a two-parameter Poisson-Weighted Akash distribution, a Poisson mixture of two-parameter weighted Akash distribution proposed by Shanker and Shukla (2016). Its moments based measures including coefficients of variation, skewness, kurtosis and index of dispersion have been derived and their natures have been discussed graphically. The estimation of parameters has been discussed using the method of maximum likelihood. Applications and goodness of fit of the distribution has also been discussed through two examples of observed real count datasets and the fit has been compared with other discrete distributions.

2. The Poisson-weighted Akash distribution

Assuming that the parameter λ of the Poisson distribution follows WAD (1.9), the Poisson mixture of WAD can be obtained as

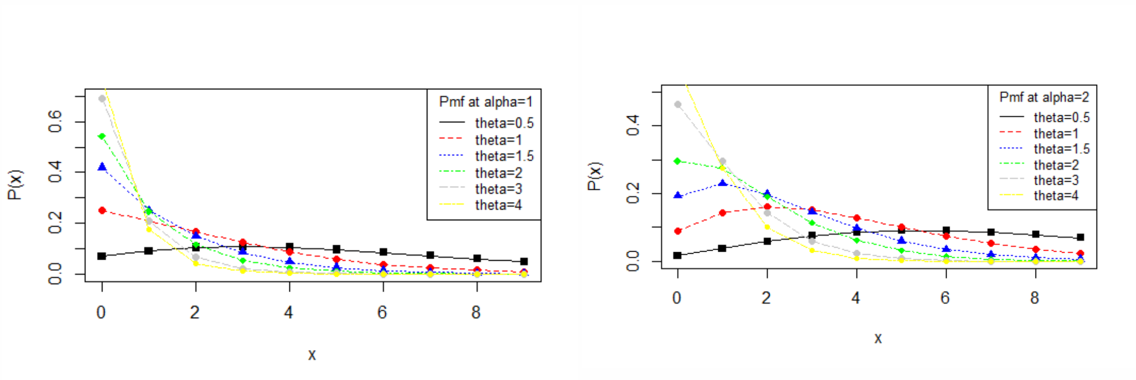
$$P_4(x;\theta,\alpha)=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)}\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1+\lambda^2)e^{-\theta\lambda}d\lambda \tag{2.1}$$

$$=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)\Gamma(x+1)}\left[\int_0^\infty e^{-(\theta+1)\lambda}\lambda^{x+\alpha-1}d\lambda+\int_0^\infty e^{-(\theta+1)\lambda}\lambda^{x+\alpha+2-1}d\lambda\right]$$

$$=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)\Gamma(x+1)}\left[\frac{\Gamma(x+\alpha)}{(\theta+1)^{x+\alpha}}+\frac{\Gamma(x+\alpha+2)}{(\theta+1)^{x+\alpha+2}}\right]$$

$$=\frac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{x^2+(2\alpha+1)x+(\theta^2+\alpha^2+2\theta+\alpha+1)}{(\theta+1)^{x+\alpha+2}};x=0,1,2,...,\theta>0,\alpha>0 \tag{2.2}$$

We would call this pmf the Poisson - Weighted Akash distribution (P-WAD). It can be easily verified that PAD (1.1) is a particular case of P-WAD for $\alpha=1$. The natures of P-WAD for varying values of the parameters θ and α have been explained graphically in figure 1. It is observed that pmf is decreasing as increased value of θ whereas pmf is decreasing as increased value of α .



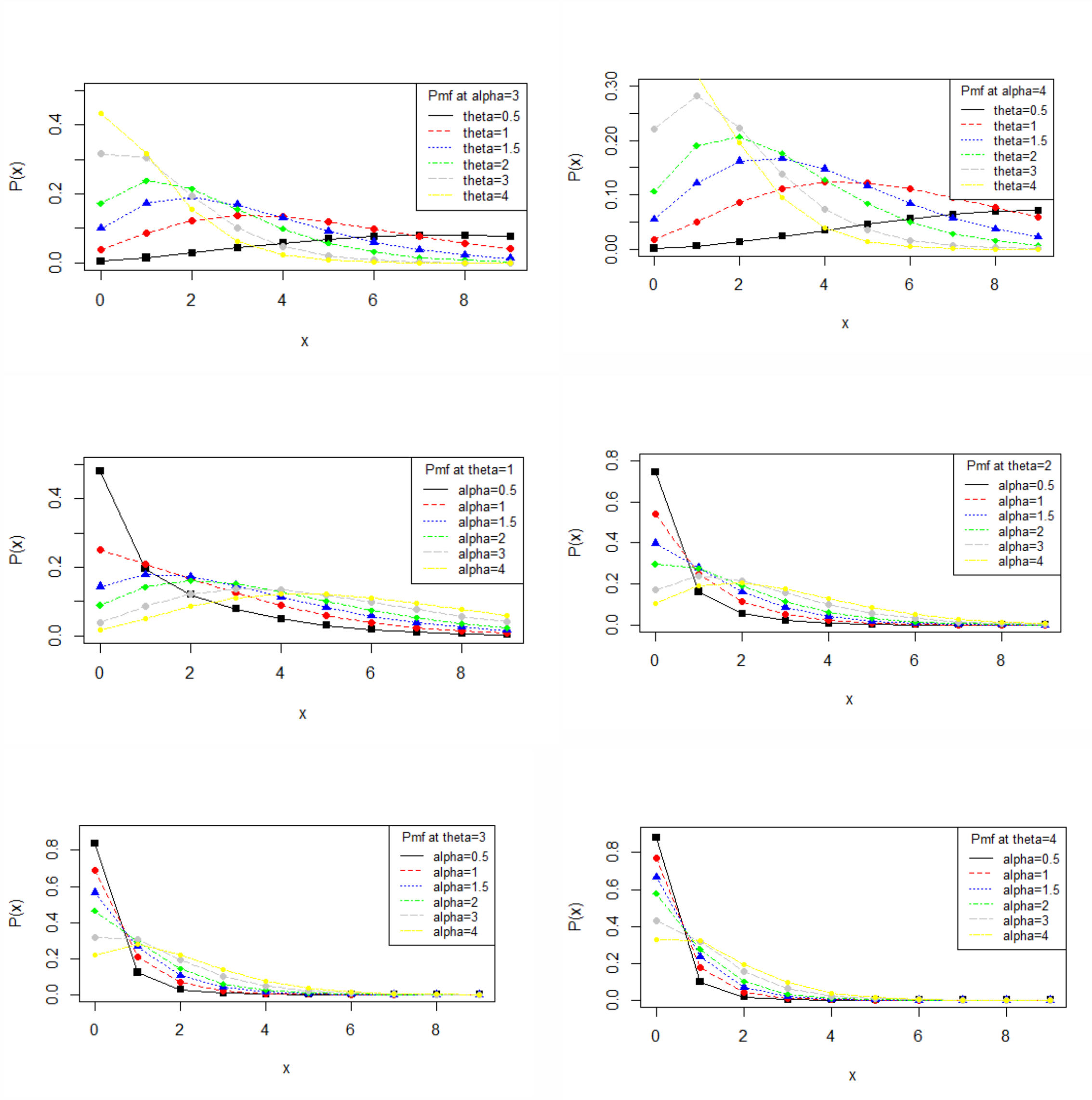


Figure 1. Probability mass function plot of P-WAD for varying values of parameters θ and α

3. Moments, skewness, kurtosis and index of dispersion

Using (2.1), the r th factorial moment about origin of the P-WAD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right], \text{ where } X^{(r)} = X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)$$

$$= \int_0^\infty \left[\sum_{x=1}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)}\right] \frac{\theta^{\alpha+2}}{\left(\theta^2+\alpha^2+\alpha\right)} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}\left(1+\lambda^2\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^{\alpha+2}}{\left(\theta^2+\alpha^2+\alpha\right)\Gamma(\alpha)} \int_0^\infty \left[\lambda^r \left\{\sum_{x=r}^{\infty} x \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\}\right] \lambda^{\alpha-1}\left(1+\lambda^2\right)e^{-\theta\lambda}d\lambda$$

Taking $x-r=y$, we get

$$\mu_{(r)}' = \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\int_0^\infty\left[\lambda^r\left\{\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}\right\}\right]\lambda^{\alpha-1}(1+\lambda^2)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha+r-1}(1+\lambda^2)e^{-\theta\lambda}d\lambda$$

$$= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\frac{\theta^2+(\alpha+r)(\alpha+r+1)}{\theta^r(\theta^2+\alpha^2+\alpha)} \quad ;r=1,2,3,.... \tag{3.1}$$

Taking $r=1,2,3,$ and 4 in (3.1), the first four factorial moments about origin of P-WAD (2.2) can be obtained

$$\mu_{(1)}' = \frac{\alpha\{\theta^2+(\alpha+1)(\alpha+2)\}}{\theta(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(2)}' = \frac{\alpha(\alpha+1)\{\theta^2+(\alpha+2)(\alpha+3)\}}{\theta^2(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(3)}' = \frac{\alpha(\alpha+1)(\alpha+2)\{\theta^2+(\alpha+3)(\alpha+4)\}}{\theta^3(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(4)}' = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\{\theta^2+(\alpha+4)(\alpha+5)\}}{\theta^4(\theta^2+\alpha^2+\alpha)}.$$

Now using the relationship between factorial moments about origin and the moments about origin, the first four moments about origin of P-WAD (2.2) can be obtained as

$$\mu_1' = \frac{\alpha(\theta^2+\alpha^2+3\alpha+2)}{\theta(\theta^2+\alpha^2+\alpha)}$$

$$\mu_2' = \frac{\alpha\{\theta^3+(\alpha+1)\theta^2+(\alpha^2+3\alpha+2)\theta+(\alpha^3+6\alpha^2+11\alpha+6)\}}{\theta^2(\theta^2+\alpha^2+\alpha)}$$

$$\mu_3' = \frac{\alpha\left\{\begin{array}{l}\theta^4+3(\alpha+1)\theta^3+2(\alpha^2+3\alpha+2)\theta^2+3(\alpha^3+6\alpha^2+11\alpha+6)\theta\\+(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24)\end{array}\right\}}{\theta^3(\theta^2+\alpha^2+\alpha)}$$

$$\mu_4' = \frac{\alpha\left\{\begin{array}{l}\theta^5+7(\alpha+1)\theta^4+7(\alpha^2+3\alpha+2)\theta^3+8(\alpha^3+6\alpha^2+11\alpha+6)\theta^2\\+6(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24)\theta+(\alpha^5+15\alpha^4+85\alpha^3+225\alpha^2+274\alpha+120)\end{array}\right\}}{\theta^4(\theta^2+\alpha^2+\alpha)}$$

Now, using the relationship $\mu_r = E\left(Y-\mu_1'\right)^r = \sum_{k=0}^{r}\binom{r}{k}\mu_k'\left(-\mu_1'\right)^{r-k}$ between central moments and the moments about origin, the central moments of the P-WAD (2.2) can be obtained as

$$\mu_2=\frac{\alpha\begin{Bmatrix}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta\\+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{Bmatrix}}{\theta^2\left(\theta^2+\alpha^2+\alpha\right)^2}$$

$$\mu_3=\frac{\alpha\begin{Bmatrix}\theta^8+3\theta^7+\left(3\alpha^2+5\alpha+4\right)\theta^6+\left(9\alpha^2+27\alpha+18\right)\theta^5+\left(3\alpha^4+10\alpha^3+17\alpha^2+34\alpha+24\right)\theta^4\\+\left(9\alpha^4+42\alpha^3+57\alpha^2+24\alpha\right)\theta^3+\left(\alpha^6+5\alpha^5+15\alpha^4+31\alpha^3+32\alpha^2+12\alpha\right)\theta^2\\+\left(3\alpha^6+15\alpha^5+27\alpha^4+21\alpha^3+6\alpha^2\right)\theta+\left(2\alpha^6+10\alpha^5+18\alpha^4+14\alpha^3+4\alpha^2\right)\end{Bmatrix}}{\theta^3\left(\theta^2+\alpha^2+\alpha\right)^3}$$

$$\mu_4=\frac{\alpha\begin{Bmatrix}\theta^{11}+\left(3\alpha+7\right)\theta^{10}+\left(4\alpha^2+12\alpha+14\right)\theta^9+\left(12\alpha^3+52\alpha^2+85\alpha+48\right)\theta^8\\+\left(6\alpha^4+42\alpha^3+138\alpha^2+246\alpha+144\right)\theta^7+\left(18\alpha^5+114\alpha^4+296\alpha^3+370\alpha^2+290\alpha+120\right)\theta^6\\+\left(4\alpha^6+54\alpha^5+294\alpha^4+730\alpha^3+774\alpha^2+288\alpha\right)\theta^5\\+\left(12\alpha^7+100\alpha^6+340\alpha^5+642\alpha^4+772\alpha^3+550\alpha^2+168\alpha\right)\theta^4\\+\left(\alpha^8+30\alpha^7+230\alpha^6+712\alpha^5+1041\alpha^4+722\alpha^3+192\alpha^2\right)\theta^3\\+\left(3\alpha^9+31\alpha^8+132\alpha^7+350\alpha^6+643\alpha^5+747\alpha^4+470\alpha^3+120\alpha^2\right)\theta^2\\+\left(6\alpha^9+60\alpha^8+228\alpha^7+432\alpha^6+438\alpha^5+228\alpha^4+48\alpha^3\right)\theta\\+\left(3\alpha^9+30\alpha^8+114\alpha^7+216\alpha^6+219\alpha^5+114\alpha^4+24\alpha^3\right)\end{Bmatrix}}{\theta^4\left(\theta^2+\alpha^2+\alpha\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the P-WAD (2.2) are thus obtained as

$$C.V=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\begin{Bmatrix}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta\\+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{Bmatrix}}}{\sqrt{\alpha}\left(\theta^2+\alpha^2+3\alpha+2\right)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{\begin{Bmatrix}\theta^8+3\theta^7+\left(3\alpha^2+5\alpha+4\right)\theta^6+\left(9\alpha^2+27\alpha+18\right)\theta^5+\left(3\alpha^4+10\alpha^3+17\alpha^2+34\alpha+24\right)\theta^4\\+\left(9\alpha^4+42\alpha^3+57\alpha^2+24\alpha\right)\theta^3+\left(\alpha^6+5\alpha^5+15\alpha^4+31\alpha^3+32\alpha^2+12\alpha\right)\theta^2\\+\left(3\alpha^6+15\alpha^5+27\alpha^4+21\alpha^3+6\alpha^2\right)\theta+\left(2\alpha^6+10\alpha^5+18\alpha^4+14\alpha^3+4\alpha^2\right)\end{Bmatrix}}{\sqrt{\alpha}\begin{Bmatrix}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta\\+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{Bmatrix}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{{\mu_2}^2}=\frac{\left\{\begin{array}{l}\theta^{11}+(3\alpha+7)\theta^{10}+\left(4\alpha^2+12\alpha+14\right)\theta^9+\left(12\alpha^3+52\alpha^2+85\alpha+48\right)\theta^8\\+\left(6\alpha^4+42\alpha^3+138\alpha^2+246\alpha+144\right)\theta^7+\left(18\alpha^5+114\alpha^4+296\alpha^3+370\alpha^2+290\alpha+120\right)\theta^6\\+\left(4\alpha^6+54\alpha^5+294\alpha^4+730\alpha^3+774\alpha^2+288\alpha\right)\theta^5\\+\left(12\alpha^7+100\alpha^6+340\alpha^5+642\alpha^4+772\alpha^3+550\alpha^2+168\alpha\right)\theta^4\\+\left(\alpha^8+30\alpha^7+230\alpha^6+712\alpha^5+1041\alpha^4+722\alpha^3+192\alpha^2\right)\theta^3\\+\left(3\alpha^9+31\alpha^8+132\alpha^7+350\alpha^6+643\alpha^5+747\alpha^4+470\alpha^3+120\alpha^2\right)\theta^2\\+\left(6\alpha^9+60\alpha^8+228\alpha^7+432\alpha^6+438\alpha^5+228\alpha^4+48\alpha^3\right)\theta\\+\left(3\alpha^9+30\alpha^8+114\alpha^7+216\alpha^6+219\alpha^5+114\alpha^4+24\alpha^3\right)\end{array}\right\}}{\alpha\left\{\begin{array}{l}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta\\+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{array}\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left\{\begin{array}{l}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta\\+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{array}\right\}}{\theta\left(\theta^2+\alpha^2+\alpha\right)\left(\theta^2+\alpha^2+3\alpha+2\right)}.$$

Behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of P-WAD for varying values of parameters θ and α have been shown graphically in figure 2.

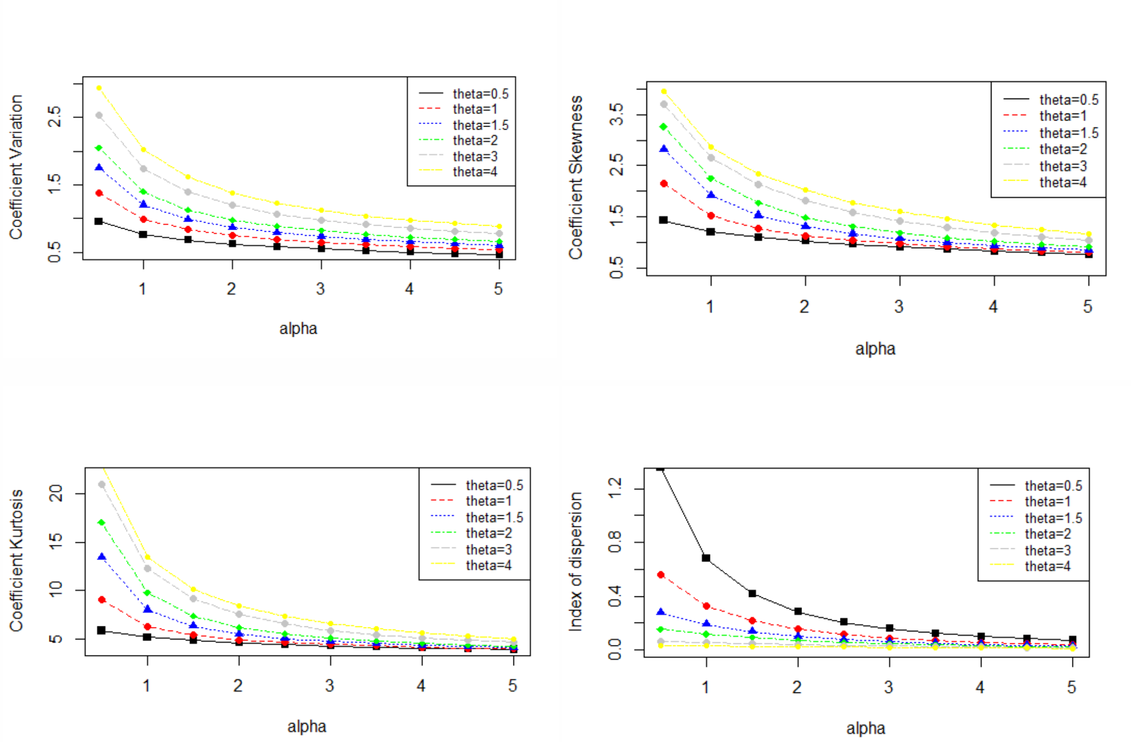


Figure 2. Behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of P-WAD for varying values of parameters θ and α

4. Maximum likelihood estimation

Let $(x_1, x_2, ..., x_n)$ be a random sample of size n from the P-WAD (2.2) and let f_x be the observed frequency in the sample corresponding to $X = x\ (x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The log likelihood function of P-WAD (2.2) can be given by

$$\log L = n\left[(\alpha+2)\log\theta - \log\left(\theta^2+\alpha^2+\alpha\right)\right] + \sum_{x=1}^{k} f_x\left[\log\Gamma(x+\alpha) - \log\Gamma(\alpha) - \log(x+1)\right]$$
$$-\sum_{x=1}^{k} f_x(x+\alpha+2)\log(\theta+1) + \sum_{x=1}^{k} f_x \log\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of P-WAD (2.2) is the solutions of the following log likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{n(\alpha+2)}{\theta} - \frac{2n\theta}{\theta^2+\alpha^2+\alpha} - \sum_{x=1}^{k}\frac{(x+\alpha+2) f_x}{\theta+1}$$
$$+\sum_{x=1}^{k}\frac{2(\theta+1) f_x}{x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = n\log\theta - \frac{n(2\alpha+1)}{\theta^2+\alpha^2+\alpha} + \sum_{x=1}^{k} f_x\left[\psi(x+\alpha) - \psi(\alpha)\right] - \sum_{x=1}^{k} f_x \log(\theta+1)$$
$$+\sum_{x=1}^{k}\frac{(2x+2\alpha+1) f_x}{x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)} = 0$$

where $\bar{x}$ is the sample mean and $\psi(x+\alpha) = \frac{d}{d\alpha}\log\Gamma(x+\alpha)$ and $\psi(\alpha) = \frac{d}{d\alpha}\log\Gamma(\alpha)$ are digamma functions. These two log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n(\alpha+2)}{\theta^2} - \frac{2n\left(\alpha^2-\theta^2+\alpha\right)}{\left(\theta^2+\alpha^2+\alpha\right)^2} + \sum_{x=1}^{k}\frac{(x+\alpha+2) f_x}{(\theta+1)^2}$$
$$+\sum_{x=1}^{k}\frac{2\left\{x^2+(2\alpha+1)x+\left(\alpha^2-\theta^2-2\theta+\alpha-1\right)\right\} f_x}{\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{-n\left\{2\left(\theta^2+\alpha^2+\alpha\right)-\left(2\alpha+1\right)^2\right\}}{\left(\theta^2+\alpha^2+\alpha\right)^2} + \sum_{x=1}^{k}\left[f_x \psi'(x+\alpha)-\psi'(\alpha)\right]$$
$$+\sum_{x=1}^{k}\frac{\left[2\left\{x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right\}-\left(2x+2\alpha+1\right)^2\right]f_x}{\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{n}{\theta} + \frac{2n\theta(2\alpha+1)}{\left(\theta^2+\alpha^2+\alpha\right)^2} - \sum_{x=1}^{k}\frac{f_x}{\theta+1} - \sum_{x=1}^{k}\frac{2(\theta+1)^2 f_x}{\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2} = \frac{\partial^2 \log L}{\partial \alpha \partial \theta}$$

where $\psi'(x+\alpha)=\frac{d}{d\alpha}\psi(x+\alpha)$ and $\psi'(\alpha+1)=\frac{d}{d\alpha}\psi(\alpha+1)$ are trigamma functions.

The maximum likelihood estimates $(\hat{\theta},\hat{\alpha})$ of (θ,α) of P-WAD (2.2) is the solution of the following equations

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

5. Applications

In this section the applications of the P-WAD has been discussed with two count datasets from biological sciences. The dataset in table 1 is the data regarding the number of European red mites on apple leaves, available in Bliss (1953). The dataset in 2 is the frequencies of the observed number of days that experienced X thunderstorm events at Cape Kennedy, Florida for the 11-year period of record in the month of June and July, January 1957 to December 1967 and are available in Falls *et al* (1971) and Carter (2001). The goodness of fit of P-WAD has been compared with the goodness of fit given by Poisson distribution (PD), PLD, PAD, and P-WLD. Note that the estimates of the parameters are based on maximum likelihood estimates for all the considered distributions. Based on the values of chi-square (χ^2), $-2\log L$ and AIC (Akaike Information criterion), it is obvious that P-WAD is competing well with the considered distributions and gives better fit. Note that AIC has been calculated using the formula $AIC=-2\log L+2k$, where k is the number of parameters involved in the distribution.

Table 1. Observed and Expected number of European red mites on Apple leaves, available in Bliss (1953)

Number of Red mites per leaf	Observed frequency	Expected frequency				
		PD	PLD	PAD	P-WLD	P-WAD
0	70	47.6	67.2	78.0	69.8	70.6
1	38	54.6	38.9	37.3	36.8	35.6
2	17	31.3	21.2	18.3	20.1	20.0
3	10	11.9	11.1	8.8	10.9	11.1
4	9	3.4	5.7	4.1	5.8	6.0
5	3	0.8	2.8	1.8	3.0	3.2
6	2	0.2	1.4	0.8	1.6	1.6
7	1	0.1	0.9	0.3	0.8	0.8
8	0	0.1	0.8	0.6	1.2	1.1
Total	150	150.0	150.0	150.0	150.0	150.0
ML estimates		$\hat{\theta}=1.14666$	$\hat{\theta}=1.26010$	$\hat{\theta}=1.89341$	$\hat{\theta}=1.09141$ $\hat{\alpha}=0.82194$	$\hat{\theta}=1.4585$ $\hat{\alpha}=0.8360$
Standard Errors		0.08743	0.11390	0.13240	0.26231 0.25230	0.12627 0.06936
χ^2		26.50	2.49	8.29	2.41	2.29
d.f		2	4	3	3	3
p-value		0.0000	0.5595	0.04038	0.4917	0.5144
$-2\log L$		485.61	445.02	447.02	425.35	439.41
AIC		487.61	447.02	449.02	429.35	443.41

Table 2. Frequencies of the observed number of days that experienced X thunderstorm events at Cape Kennedy, Florida for the 11-year period of record in the month of June, January 1957 to December 1967

X	Observed frequency	Expected frequency				
		PD	PLD	PAD	P-WLD	P-WAD
0	187	155.6	185.3	190.7	185.1	187.6
1	77	116.9	83.4	79.7	83.7	80.5
2	40	43.9	35.9	34.4	36.0	35.4
3	17	11.0	15.0	14.7	15.0	15.4
4	6	2.0	6.1	6.1	6.1	6.5
5	2	0.3	2.5	2.5	2.4	2.7
6	1	0.3	1.8	1.9	1.7	1.9
Total	330	330.0	330.0	330.0	330.0	330.0
ML estimate		$\hat{\theta}=0.75148$	$\hat{\theta}=1.80427$	$\hat{\theta}=2.17976$	$\hat{\theta}=1.82188$ $\hat{\alpha}=1.01237$	$\hat{\theta}=2.15124$ $\hat{\alpha}=1.01198$
Standard Errors $\hat{\theta}$ $\hat{\alpha}$		0.04772	0.12573	0.10781	0.41748 0.28219	0.13789 0.05056
χ^2		31.6	1.43	1.64	1.41	1.31
d.f		2	3	3	2	2
p-value		0.0000	0.6985	0.6503	0.4941	0.5194
$-2\log L$		824.50	788.88	840.66	874.20	788.84
AIC		826.50	790.88	842.66	878.20	788.73

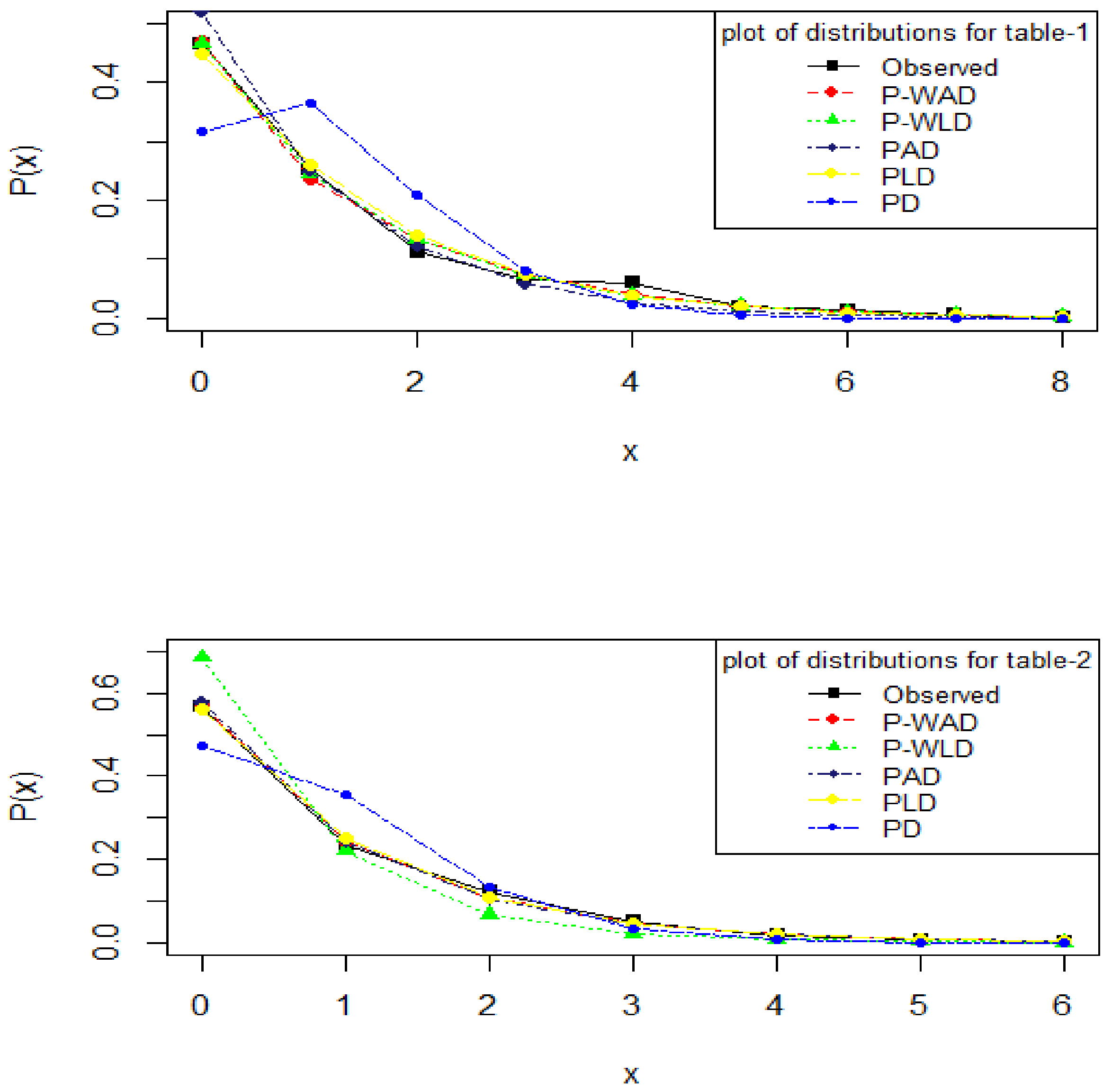


Figure 3. Fitted probability plots for distributions

6. Concluding remarks

A Poisson-Weighted Akash distribution which includes Poisson-Akash distribution has been proposed. Its moments and moments based statistical constants have been derived and studied. Some statistical properties have been discussed. Maximum likelihood estimation has been discussed for estimating parameters of the distribution. Finally, applications of the proposed distribution have been explained through some count datasets and the goodness of fit has been compared with other discrete two parameter and one parameter distributions and it was found satisfactory over P-WLD, PAD, PLD, and PD on considered data sets.

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