HEURISTIC DECISION MAKING UTILIZING COMPLETE TOURNAMENTS[1]

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Abstract

The paper presents the newly developed Abarnica heuristic ranking method applied to fuzzy-bias decision making with the central technique derived from complete tournaments or derivatives thereof. An useful recursive proposition called, Akhil's proposition together with a challenging conjecture is presented. The challenge to derive an efficient algorithm to apply the Abarnica heuristic method which appears to be very efficient with manual applications remains open. The authors advocate that the main advantage of the Abarnica heuristic ranking method is that strong bias are analytically mitigated in fuzzy-bias decision making applications which require ranking.

Keywords: *Complete tournament, Perron-Frobenius theorem, Abarnica heuristic, corrupt driving testing*

1. Introduction

For general notation and concepts in graphs and digraphs see [1, 2, 3]. Unless mentioned otherwise all graphs are simple, connected and directed graphs (digraphs). Furthermore, if the context is clear the terms *vertex* and *player* will be used interchangeably.

A directed complete graph of order $n \geq 1$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ can be considered to be complete tournament, denoted by T_n. The understanding of a complete tournament is that all distinct pairs of vertices are players in a match and the arc (v_i, v_j)

indicates that player v_i beat player v_j. Unless mentioned otherwise a match between a distinct pair of players does not allow a draw. In other words it is assumed that an arc has distinct orientation. The assumption can easily be relaxed to analyze bi-orientations as well. Figure 1 depicts a complete tournament, T_6

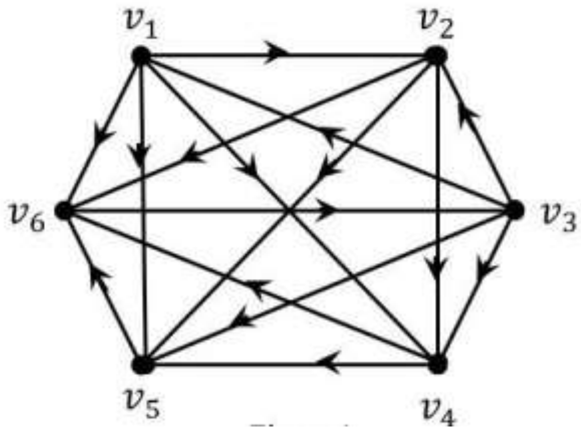

Figure 1.

If a vertex $v_i \in V(T_n)$ exists with out-degree $d^+(v_i) = n-1$, vertex v_i is the outright winner of the tournament. It is easy to see only one such vertex can exist. Also, if a vertex $v_j \in V(T_n)$ exists with in-degree, $d^-(v_j) = 0$, vertex v_j is the outright loser of the tournament. Only one such vertex can exist. For further analysis we shall only consider the subgraph $T - \{v_i, v_j\} = T^{'}_{n-2}$. The aforesaid is called the *reduction rule*. The reduction rule is applied iteratively until no further reduction is possible. Therefore, the final graph is either the empty graph hence, $V(T_0^{'''...'}) = \phi$, if *n* is even, or $T_1^{'''...'} = K_1$ if *n* is odd, or $T_k, k \geq 3$ results. Clearly for the first two outcomes the corresponding *trivial ranking* of the tournaments have been derived.

2. Important Graph Theoretical Results of Complete Tournaments

Unless mentioned otherwise, a resultant complete tournament say, $T_n, n \geq 3$ will be assumed. Figure 1 depicts a complete tournament that cannot be reduced further. It is assumed that the reader is familiar with the concept of a directed path and distance between vertices v_i and v_j. A *king vertex* v_i is a vertex for which the maximum directed distance between v_i and v_j, $\forall j \neq i$ is 2. For a directed path between vertices v_i and v_j of length 2 it is said that v_i gained a *virtual win* over v_j. Note that the aforesaid virtual win is possible despite a direct win of v_j over v_i.

Lemma 2.1 Without applying the reduction rule, a complete tournament T_3 has either one or three king vertices.

Proof: Up to isomorphism the only complete tournaments T_3 that exist are

$T_3 = (V(T_3), A(T_3)) = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\})$ or

$T_3 = (V(T_3), A(T_3)) = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\})$

Hence:

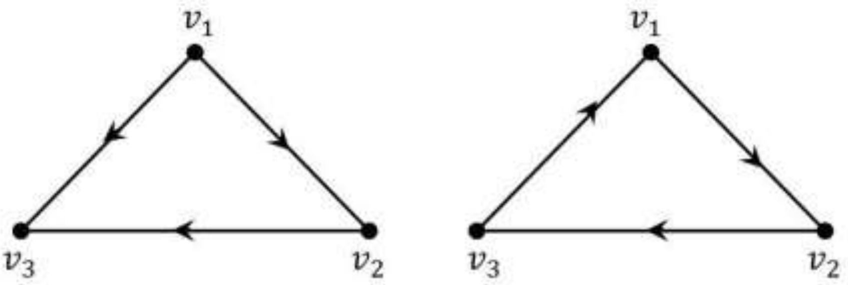

Figure 2.

Therefore, either only one king vertex, v_1 or three king vertices, v_1, v_2, v_3 exist.

Lemma 2.1 implies that for the case of having one king vertex the final ranking is $v_1, v_2\ v_3$

For the case of three king vertices the ranking requires at least one further match rule and/or at least one further heuristic criteria to resolve the ties. An example of heuristic criteria could be that player (vertex) v_1 displayed superior technique compares to that of players v_2, v_3 hence, the win of v_3 was a fluke. The heuristic criteria can be applied by say, a panel decision to break the tie between v_1 and v_3 or a re-match (match rule) is allowed.

A resultant complete tournament (reduction rule applied) is said to be diconnected if for any two distinct vertices, v_i, v_j there exists a directed path from v_i to v_j and from v_j to v_i.

Lemma 2.2 (Formalisation of observation in [1]) Up to isomorphism, there exists only one diconnected tournament of order 4.

Proof: Through exhaustive combinatorial re-orientation of arcs the figure below proves the result.

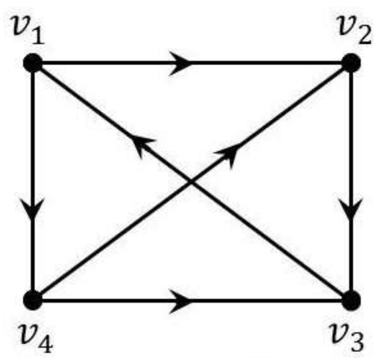

Figure 3.

Lemma 2.2 implies that all complete tournaments of order 4 other than the diconnected complete tournament can be trivially ranked by applying the reduction rule together with Lemma 1.1. From [1, pp. 185-188] it follows that all resultant complete tournaments on $n \geq 5$ vertices are disconnected and can be ranked by the Perron-Frobenius theorem.

2.1 The Abarnica Heuristic Ranking Method

The proposed Abarnica Heuristic Ranking Method[3] utilises the random selection of a sufficient number of primitive holes (see [4]) of a complete graph on $n \geq 4$ vertices to obtain only T_3 complete tournaments. These are then assembled to construct the complete tournament, T_n. This method has the advantage that different panels of judges or assessors can be utilised for the different T_3 tournaments when the rules are fuzzy-bias. Examples of such fuzzy-bias decision making tournaments are beauty contests, ranking political candidates, ranking music genres, ranking food brands, expert estimation of the impact speeds based on the crash damage to vehicles, agreeability within societies and alike. Hence, in decision making tournaments where the decision (game) rules are not a reliable approximation of objectivity.

Illustration 1. Consider the complete tournaments:

$T_3^1 = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_3, v_2), (v_3, v_1)\})$, $T_3^2 = (\{v_1, v_3, v_4\}, \{(v_3, v_1), (v_3, v_4), (v_4, v_1)\})$,

$T_3^3 = (\{v_1, v_3, v_5\}, \{(v_1, v_3), (v_5, v_3), (v_1, v_5)\})$, $T_3^4 = (\{v_1, v_5, v_6\}, \{(v_1, v_5), (v_5, v_6), (v_1, v_6)\})$,

$T_3^5 = (\{v_2, v_4, v_5\}, \{(v_2, v_4), (v_2, v_5), (v_4, v_5)\})$, $T_3^6 = (\{v_2, v_5, v_6\}, \{(v_2, v_5), (v_2, v_6), (v_5, v_6)\})$,

$T_3^7 = (\{v_3, v_5, v_6\}, \{(v_5, v_3), (v_6, v_5), (v_6, v_3)\})$, $T_3^2 = (\{v_4, v_5, v_6\}, \{(v_4, v_5), (v_4, v_6), (v_5, v_6))$.

Since the number of panels and the stratification of the panel members can be arbitrary or well-structured (experts) we assume four panels namely P_1, P_2, P_3, P_4 provided the outcome above. Note that no panel could give absolute bias preference to any particular player (vertex). Also note that the tournaments were selected randomly. The only requirement to be met is that assembly into a singular tournament T_6 must result in a complete tournament with no tie in the orientation of an arc. In [4] it was shown that the number of primitive holes of a complete graph K_n is given by, $\binom{n}{3}$. It is easy to see that the assembly of the eight $T_3^i, i = 1,2,3,...,8$ tournaments results in the complete tournament depicted in figure 1. The observation that the complete tournament can be assembled by eight triangular tournaments compared to the twenty (20) distinct primitive holes found in K_6 signals great efficiency. In fact assembling is possible with only six (6) carefully selected primitive holes. Let $H(K_n)$ denote the minimum number of primitive holes of K_n to be oriented such that a complete tournament T_n can be assembled. In the next result the expression,

$\left\lfloor \frac{n}{2} \right\rfloor + 1$ is deliberately preferred above the simpler expression, $\left\lceil \frac{n}{2} \right\rceil$. The motivation is that the first expression most likely relates to the challenge to prove Conjecture 2.3.1.

Proposition 2.3 (Akhil's Proposition)[4] Any complete graph $K_n, n \geq 4$ has a minimum of $H(K_n) \leq H(K_{n-1}) + \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right)$ primitive holes (K_3) (not necessary unique) for which the complete tournaments, $T_3^i, i = 1,2,3,\ldots, H(K_n)$ on appropriate assembling, result in a complete tournament T_n (not necessary diconnected), provided that no arc orientation tie exists.

Proof: We begin by considering the complete tournament, T_3. For both of the only possible complete tournaments T_3^1 and T_3^2 it is clear that a mixed complete graph is obtained if vertex v_4 and edges v_1v_4, v_2v_4, v_3v_4 are added. Also, a minimum of two primitives holes say, on vertices v_1, v_2, v_4 and v_1, v_3, v_4 or v_1, v_2, v_4 and v_2, v_3, v_4 or v_1, v_3, v_4 and v_2, v_3, v_4 must be oriented. Thereafter, appropriate assembling any of the combinations will result in a complete tournament, T_4. Hence, the result $H(K_4) = 3 \leq H(K_3) + \left\lfloor \frac{3}{2} \right\rfloor + 1$ holds.

Assume the result holds for $n = k$ and consider any complete tournament, T_n.

Case 1: Let k be even and add vertex v_{k+1} and the edges, $v_1v_{k+1}, v_2v_{k+1}, v_3v_{k+1}, \ldots, v_kv_{k+1}$. Therefore, the maximum additional primitive holes to be orientated is, $\frac{k}{2}$. Since,

$H(K_{k+1}) \leq H(K_k) + \frac{k}{2} < H(K_k) + \left\lfloor \frac{k}{2} \right\rfloor + 1$, the result holds $\forall n$ is even.

Case 2: Let k be odd and add vertex v_{k+1} and the edges, $v_1v_{k+1}, v_2v_{k+1}, v_3v_{k+1}, \ldots, v_kv_{k+1}$. Therefore, the maximum additional primitive holes to be orientated is, $\left\lfloor \frac{k}{2} \right\rfloor + 1$. Since,

$H(K_{k+1}) \leq H(K_k) + \left\lfloor \frac{k}{2} \right\rfloor + 1$, the result holds $\forall n$ is odd.

Hence, through induction it follows that the results holds, $\forall n \geq 4$.

Conjecture 2.3.1 A minimum number of primitive holes, $H(K_n) \leq \left\lfloor \frac{\varepsilon(K_n)}{3} \right\rfloor + 1$ of a complete graph $K_n, n \geq 4$ can be orientated such that a complete tournament T_n can be assembled.

Consider figure 3 and note that vertices v_1, v_3, v_4 are king vertices. Also note that more than one distinct king path (directed path of length at most 2) may exist between vertices v_i and v_j.

Definition 2.1 The king index of a vertex v_i is the sum of the number of distinct king paths from v_i to v_j, $\forall j$. The king index is denoted, $k(v_i) \to V(T_n) - \{v_i\} = l, l \in N$.

Illustration 2. From figure 3 it follows that: $k(v_1) \to \{v_2, v_3, v_4\} = 2+2+1=5$,

$k(v_2) \to \{v_1, v_3, v_4\} = 1+1=2,$

$k(v_3) \to \{v_1, v_2, v_4\} = 1+1+1=3$ and

$k(v_4) \to \{v_1, v_2, v_3\} = 1++2=4$.

Since no king index ties exist the final ranking follows easily as: v_1, v_4, v_3, v_2.

Illustration 3. From figure 1 it follows that:

$k(v_1) \to \{v_2, v_3, v_4, v_5, v_6\} = 1+2+2+3+4=12,$

$k(v_2) \to \{v_1, v_3, v_4, v_5, v_6\} = 2+1+2+3=8,$

$k(v_3) \to \{v_1, v_2, v_4, v_5, v_6\} = 1+2+3+3+3=12,$

$k(v_4) \to \{v_1, v_2, v_3, v_5, v_6\} = 2+1+2=5,$

$k(v_5) \to \{v_1, v_2, v_3, v_4, v_6\} = 1+1+2+1+1=6$ and

$k(v_6) \to \{v_1, v_2, v_3, v_4, v_5\} = 1+1+1+1=4$.

The tie between vertices v_1, v_3 is resolved by noting that v_1 has king index summation term of 2 (total virtual wins) in respect of v_3 whilst v_3 has king index summation term of 1 (total virtual wins) in respect of v_1. Therefore, the derived ranking is, $v_1, v_3, v_2, v_5, v_4, v_6$. It is noted from [1] that the ranking derived is equal to the ranking derived by the Perron-Frobenius theorem.

Summarizing the Abarnica Heuristic Ranking Method

Step 1: Consider the complete graph $K_n, n \geq 4, n \in N$ and select at least $\left\lfloor \frac{\varepsilon(K_n)}{3} \right\rfloor + 1$ distinct primitive holes such that K_n can be assembled from these primitive holes.

Step 2: Choose a finite number of panels to decide on the orientation of the edges where an arc (v_i, v_j) will imply that vertex v_i is preferred, or is the winner over vertex v_j in terms of game rules or fuzzy-bias decision criteria.

Step 3: Assemble the complete tournament and apply the reduction rule to derive the preliminary ranking of all outright winners and losers. If the reduction rule results in a trivial ranking, exit. Else, go to Step 4.

Step 4: Determine the king index of all vertices of the resultant diconnected complete tournament and derive the complete ranking through the decreasing ordering of the king indices. If a tie occurs amongst king indices go to Step 5. Else, exit.

Step 5: For a king index tie between say, v_i and v_j, compare the virtual win score of v_i versus v_j as well as the virtual win score of v_j versus v_i. The highest virtual win score wins the ranking position. If a further tie occurs go to Step 6. Else, exit.

Step 6: If a virtual win score tie occurs between say v_i and v_j, the direct winner indicated by arc say, (v_j, v_i) is used to allow player v_j to win the ranking position.

3. Application to Driving License Testing within the South African Context

In the South African context the National Road Traffic Act, (Act 93 of 1996), provides for three clusters of driving licenses i.e. Code A1, A (motorbikes); B or EB (light motor vehicles) and Code C1, C, EC1 or EC (heavy motor vehicles). In section 3.2 of [5] a noticeable imbalance in the ratio of Code C1 license holders versus the population requiring that driving license code was observed. This observation raises the mysterious question for which a plausible answer should be researched. Why is it that so many decision makers and DLTC's and driving schools claim the existence of a high demand for Code C1 driving license testing for a vehicle category < 4, 1% of the vehicle population requiring that specific driving license code? The respected digital research company, Pondering Panda, released a survey in May 2013 in which it reliably reported that corrupt driving license testing is rife in South Africa. In particular, Code C1 is of interest because anecdotal evidence suggests strongly that Code C1 driving testing is the most corrupt driving testing code in the South African context. The aforesaid observation will be tested empirically by applying the Abarnica Heuristic Ranking Method to the complete tournament defined in the next section.

3.1. Complete Tournament Structure and Fuzzy-Bias Decision Making Rule

The complete tournament has players (vertices):

$$v_1 = \{A1, A\}, v_2 = \{B, EB\}, v_3 = \{C1\}, v_4 = \{C\}, v_5 = \{EC1\}, v_6 = \{EC\}.$$

The primitive holes to be orientated are on the set of vertices: $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_2, v_6\}$, $\{v_3, v_4, v_5\}$ $\{v_3, v_4, v_6\}$, $\{v_3, v_5, v_6\}$, $\{v_1, v_3, v_6\}$, $\{v_2, v_3, v_5\}$, respectively. The primitive holes were selected to ensure good empirical scrutiny of the Code C1 driving license. The selection allows for possible orientation ties between v_3 and all other players as well an orientation tie between all pairs amongst players v_2, v_3 and v_6.

The fuzzy-bias decision making rule is: Arc (v_i, v_j) = 1 implies that driving license testing in respect of Code(s) $\in v_j$ is least corrupt (or less prone to corrupt testing practices). Responses (decision making for orientation) will be captured in the table through 0 or 1 entries to mean that $(v_i, v_j) = 1 \Leftrightarrow (v_j, v_i) = 0$. Note that populating the complete table will serve as a statistical measure of consistency since a respondent may unintentionally or through subjective thinking map both the ordered pairs $(v_i, v_j) \to 1$ and $(v_j, v_i) \to 1$. As stated earlier, bi-orientations can be analyzed. Bi-orientation only nullifies a direct win but certainly contributes to the king index as well as the virtual win score as well. See the next illustrative table.

Table 1	v_i	v_j	v_k
v_i	-	0	1
v_j	1	-	1
v_k	0	0	-

3.2. Data Collection and Analysis

A total of 750 primitive holes were circulated for evaluation. A total of 665 responses were received hence, 1995 pairs of edges were evaluated (matched). Therefore, data collection had a response ratio of 88,67% which is considered satisfactory. Interesting to note is that 70,59% of non-respondents are from amongst practising examiners from the provinces, Western Cape, KwaZulu-Natal, Gauteng and Limpopo. The final tournament outcome is depicted in Table 2 below.

Table 2	A1/A	B/EB	C1	C	EC1	EC
A1/A	-	32	18	37	0	102
B/EB	88	-	62	61	35	20
C1	102	178	-	107	99	67
C	83	79	13	-	32	62
EC1	100	185	41	68	-	77
EC	13	100	33	78	23	-

From table 2 it is clear that a zero-row does not exist. It implies that no code category is viewed as corrupt free in respect of driver testing. It is observed that the Codes A1/A are viewed as corrupt free only in comparison of the heavy vehicle driving license code namely, EC1. Table 2 supports the anecdotal evidence that Code C1 is most prone to high levels of corrupt testing in that Code C1 has the highest average row score of 94,2. Table 3 defines the complete tournament and figure 5 depicts the complete tournament.

Table 3	A1/A	B/EB	C1	C	EC1	EC
A1/A	-	0	0	0	0	89
B/EB	56	-	0	0	0	0
C1	84	116	-	94	58	34
C	46	18	0	-	0	0

EC1	100	150	0	36	-	54
EC	0	80	0	16	0	-

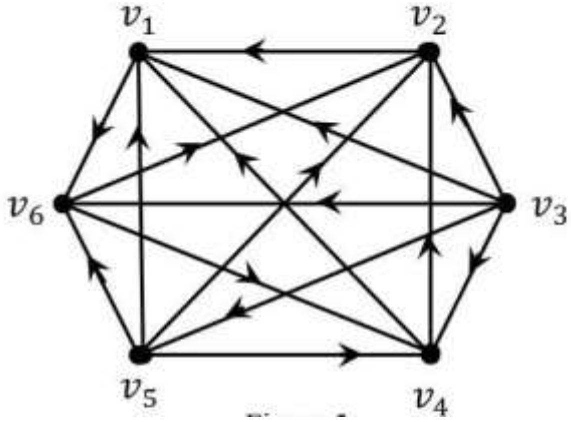

Figure 5.

Clearly vertex v_3 is the outright winner and upon reduction it follows that vertex v_5 is the second runner-up. Upon the second iterative reduction the resultant complete tournament on vertices v_1, v_2, v_4, v_6 is isomorphic to figure 3. Therefore, the final ranking is immediate i.e. $v_3, v_5, v_6, v_4, v_1, v_2$. The fact that vertex v_1 beats vertex v_2 comes somewhat as a surprise to the authors intuitive thinking. More so, since the average row score for Code A1/A is the lowest at 37,8. This result in itself requires some further investigation. A question that comes to mind is whether the growing popularity of superbikes and touring classics and/or the expanding scooter delivery and courier industry have resulted in a high demand for Codes A1 and A driving licenses.

4. Conclusion and Further Research

The application of the Abarnica Heuristic Ranking Method to the data indicates undoubtedly that the driving license testing for Code C1 can be considered as the most corrupt testing code within the South African context. The popularity of the Code C1 driving license probably lies in the fact that the driving test protocol is easier than that prescribed for a light motor vehicle. It is the authors considered view that most applicants who test for Code C1 actually do so with the intent to drive light motor vehicles. This view is currently the subject of research. However, mastering the easier driving competency might still be a challenge to many and therefore, the demand of corrupt testing and perhaps in many instances, no testing at all, to obtain a Code C1 is high. The aforesaid is then followed up by the Code EC1. The relationship between the first two driving license codes is that Code C1 is the lightest in the heavy vehicle cluster and Code EC1 is the lightest in the articulated heavy vehicle cluster. In terms of average row scores Codes B/EB and C performed in a close-tie with average row scores of 53,2 and 53,8 respectively. In [5] it was found that Code C is most likely becoming a redundant driving license code. It is suggested that the close-tie will be interesting to research further.

The power of the Abarnica Heuristic Ranking Method lies in the fact that the assessment of primitive holes mitigates strong bias or preference over a particular player (vertex) whilst the assembling of the primitive holes represents a complete tournament. The fact that Code A1/A beats Code B/EB and loses against Code C whilst Codes B/EB and C are in a close-tie illustrates the power of the Abarnica Heuristic Ranking Method to rank fuzzy-bias decision making applications.

It is clear that Akhil's proposition is a recursive result. It will be of interest to seek a closed formula and an efficient algorithm to find the minimum number of primitive holes $H(K_n)$. Any two primitive holes which do not share a common edge are said to be independent. Let the maximum number of independent holes of a graph G be denoted by, $\alpha^p(G)$. We propose an improved conjecture for further research.

Conjecture 4.1 For a complete graph K_n, $n \geq 3$, the minimum number of primitive holes of K_n to be oriented such that a complete tournament T_n can be assembled is given by

$$H(K_n) = \alpha^p(K_n) + \left\lceil \frac{\varepsilon(K_n) - 3\alpha^p(K_n)}{2} \right\rceil$$

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[3] The first author dedicates this heuristic ranking method to young lady Abarnica, the niece to the second author.

[4] The first author dedicates this proposition to young lad Akhil, the nephew to the second author.