A SPREADSHEET SOLUTION TO H. E. DUDENEY'S PUZZLE "VISITING THE TOWNS"

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Abstract:
The purpose of this paper is to present an optimization model approach to one of Dudeney's puzzles, "Visiting the Towns," which appears as puzzle number 243 in Amusements in Mathematics (Dudeney, 1917). Henry Ernest Dudeney (1857-1930) was one of the most prolific mathematical puzzle creators of the past two hundred years. His first collection of puzzles, The Canterbury Puzzles and Other Curious Problems, was published in 1907. This popular book is still in print today (Dudeney, 1907). Beginning in 1917, he created five other popular puzzle collections, three of which were published posthumously. In addition to prodigious quantities of puzzles, his range of puzzle-types was exceedingly large, including geometry-based, logic puzzles , combinatorics" (Bremner, 2011), "cryptarithmetic" (Kilpelainen, 2012), cross-number puzzles (Sit, 1991), and chess problems (Nowlan, n.d.). During his time, there were no calculators; his mathematical puzzles were approached through pencil and paper calculations.

Key words: spreadsheet solution, H. E. Dudeney, puzzle, Visiting the Towns

Introduction

Dudeney had little in the way of formal education, a fact that launches Bremner (2011) into an analysis of Dudeney's methods for finding exact solutions. For instance, the puzzle of the Silver Cubes has as its solution (x,y) = (104940/40831, 11663/40831). Although his earliest efforts appeared in periodicals, including *The Weekly Dispatch, The Queen, Blighty, Cassell's Magazine* and *The Strand* (Henry Ernest Dudeney, n.d.), his column in *The Strand,* named "Perplexities," ran for over thirty years (Nowlan, n.d.). Many of these puzzles were published under the pseudonym "Sphinx."

A brief literature review follows this section. Next, the puzzle is given in its original format. A section describing the development of the model, along with a detailed description of the model proper is next. We conclude with a discussion and summary.

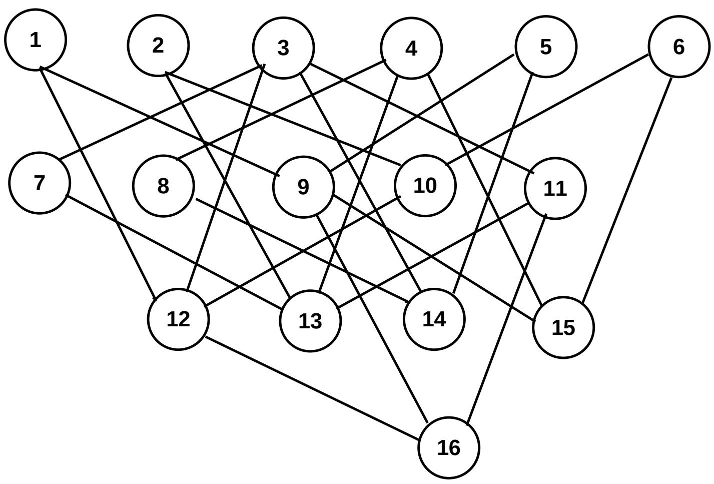
Literature Review

In addition to Dudeney, some of the better known writers in the field of recreational mathematics include Sam Loyd, Raymond Smullyan, Martin Gardner and Charles Lutwidge Dodgson, better known as Lewis Carrol, author of *Alice's Adventures in Wonderland*. "Recreational mathematics" is the term frequently used to describe mathematical games, puzzles and riddles. The best known peer-reviewed journal devoted to recreational mathematics is the *Journal of Recreational Mathematics* (Baywood Publishing, Inc., n.d.). Many journals give some passing attention to the subject, often through dedicated columns. For example, *Communications of the ACM* regularly publishes a column named "last byte" (Winkler, 2012). Alexander Dewdney (not to be confused with H. E. Dudeney!) wrote a "famous section" in *Scientific American* during the 1980s, as did Gardner for over twenty-four years prior. Dewdney's column was named "Computer Recreations" while Gardener's column was named "Mathematical Games" (Jimenez & Munoz, 2011). The column "Classroom Capsules" appears in *The College Mathematics Journal* and a brief survey shows that problems and puzzles often appear in the column for the purpose of providing "effective teaching strategies for college mathematics instruction (Alfaro, 2008)." As early as 1923, Carver suggested using Dudeney's puzzles as "stimulus" to undertake investigation, for both students and teachers!

In defining "recreational programming," Jimenez and Munoz (2011) refer to the practice as one of studying computer programming by solving problems of a playful nature. They describe the discipline as "similar to recreational mathematics." Demain (2010) uses the term "recreational computer science." Kino and Uno (2012) briefly discuss the incorporation of computers into the study of games and puzzles, and the reasons therefore. These authors modelled the game Tantrix using an interger programming formulation and solved it with an IBM software product. Kilpelainen (2012) uses problems from recreational mathematics, including some of Dudeney's other puzzles, to assess the features and utility of a new programming language. He notes that the process of formulating puzzle solution algorithms for computation invites programmers to consider problem generalization.

The Problem: Visiting the Towns (Dudeney, 1917)

243—VISITING THE TOWNS



"A traveller (sic), starting from town No. 1, wishes to visit every one of the towns once, and once only, going only by roads indicated by straight lines. How many different routes are there from which he can select? Of course, he must end his journey at No. 1, from which he started, and must take no notice of cross roads, but go straight from town to town. This is an absurdly easy puzzle, if you go the right way to work."

A Model of Visiting the Towns

Operations researchers will recognize the similarity between Dudeney's puzzle and the much researched Traveling Salesman Problem (TSP). Lawler, Lenstra, Kan, & Shmoys (1985) trace the origins of the TSP to the German publication *Der Handlungsreisende, wie er sein soll und was er zu thun hat, urn Aufträge zu erhalren und eines gluckichen Erfolgs in seinen Geschäften gewiss zu sein Von einem alten Comis-Voyageur* (Voight, 1831; Muller-Merbach, 1983). The Visiting the Towns puzzle is similar to the TSP in two ways. First, the tour must end in the same city where it begins. In this case the origin is the town labelled number "1." Second, each city must be visited only once.

The major difference between this problem and the TSP deals with the objective of the TSP, which is to minimize the total length of the journey (called a "tour"). Dudeney's problem does not identify the distance between the towns which are connected by roads. Thus, there is no objective to minimize the tour length. Instead, the objective is to designate a tour that begins in town "1," ends in town "1," and enters each town once. The general TSP does not constrain the solution space by specifying allowable roads as Dudeney's puzzle does (Press, Flannery, Teukolsky, & Vetterling, 1988).

Model Formulation of the Visiting the Towns Puzzle

The initial development of this spreadsheet model required the use of the Solver add-in tool for Excel. Solver is an optimization software tool developed by Frontline Systems and is available with Excel. During initial development, two characteristics of this problem presented particularly challenging issues. First, the model requires a constraint known as "alldifferent," which is particularly powerful in combinatorial problems, such as the TSP and, in this case, the Visiting the Towns puzzle. (Frontline Systems, n.d.). Secondly, Solver provides a genetic algorithm approach to optimization for non-smooth models and proved to be essential in the successful formulation of this model in order to arrive at an optimal solution.

The initial Solver formulation of the problem is shown in Table 1. Row 2 and column B, both highlighted, are used to identify each of the towns, labeled 1 to16. Cells C3:R18 identify those towns that are connected by a road, thus making a connection between those towns possible. A "1" indicates the presence of a connecting road. Likewise, a "0" indicates the absence of a road. For example cell D3, holding a "0," indicates that the towns 1 and 2 are not connected. Cell K1, with a value of "1" indicates a road connection between towns 1 and 9. Cells S2:S17, initially set to values 1-16, will hold the Excel-developed sequence when optimization is complete. Column U, labeled PATH, will be utilized to assure that connections are made between towns which have direct road connections. Cells C3:R18, which as previously discussed define the feasible routes between cities, is named MATRIX in Excel. While the model does not deal with distances between towns, as would be the case in a TSP, model formulation requires an objective. The defined objective in the problem is to maximize the distance of the tour. In reality, what is accomplished is the assurance that only values of 1 are included in the tour, meaning that only feasible connections between towns are used. The requirement that each city be visited only once is addressed with the “alldifferent” constraint. The Solver parameters are displayed in Figure 2.

Table 1. Initial Spreadsheet Model Formulation Visiting the Towns Puzzle

1\A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
2		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	Path
3	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	2	0
4	2	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	3	0
5	3	0	0	0	0	0	0	1	0	0	0	1	1	0	1	0	0	4	0
6	4	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	5	0
7	5	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	6	0
8	6	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	7	0
9	7	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	8	0
10	8	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	9	0
11	9	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	10	0
12	10	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	11	0
13	11	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	12	0
14	12	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	13	0
15	13	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	0	14	0
16	14	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	15	0
17	15	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	16	0
18	16	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	0
19																		Total	0
20																		Test	1

The formula view of the model is displayed in Table 2. The INDEX function returns the value in the cell at the intersection of a particular row and column. The genetic algorithm /evolutionary solver proposes a tour in column S. Since allowable connections (i.e. the roads) are worth 1 while unallowable connections are worth 0, Solver is programmed to increase the value of the resulting tour where the value is the sum of the connection values. This forces the solution to contain allowable connections since the objective is maximization. The “alldifferent” constraint forces the solution to contain only one visit to each town. The solution output from Solver is displayed in Table 3. As previously discussed, column S defines the suggested tour to travel from location 1 to 16, entering each once and only once. The sequence is 1-12-16-11-3-7-13-2-10-6-15-4-8-14-5-9-1. Figure 3 displays a flow diagram of the optimal tour. The astute observer will identify that the inverse tour (1-9-5-14-8-4-15-6-10-2-13-7-3-11-16-12-1) is also optimal. As Dudeney (1917) points out, this path (and its inverse) represent the only optimal solution to this puzzle. Solver required approximately 40 seconds to formulate the optimal answer with no adjustment to the default settings.

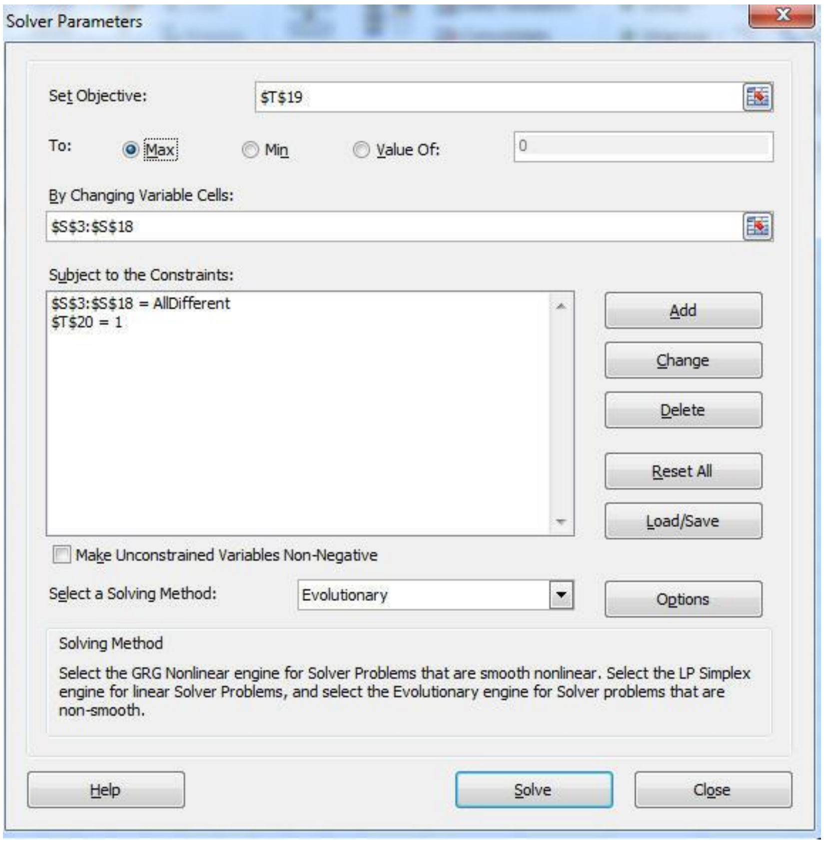


Figure 2. Solver Parameters Visiting the Towns Puzzle

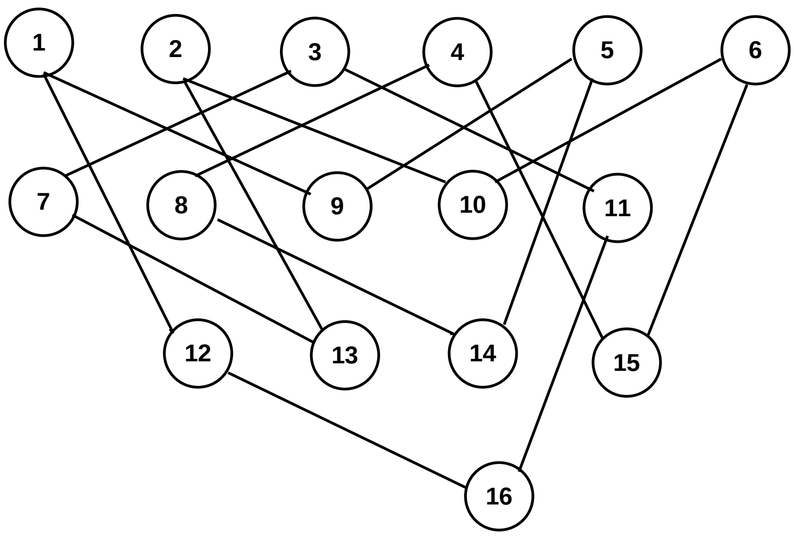
Table 2. Formula View Visiting the Towns Puzzle

1\A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
2		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	Path
3	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	2	=INDEX(MATRIX,S2,S3)
4	2	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	3	=INDEX(MATRIX,S3,S4)
5	3	0	0	0	0	0	0	1	0	0	0	1	1	0	1	0	0	4	=INDEX(MATRIX,S4,S5)
6	4	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	5	=INDEX(MATRIX,S5,S6)
7	5	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	6	=INDEX(MATRIX,S6,S7)
8	6	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	7	=INDEX(MATRIX,S7,S8)
9	7	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	8	=INDEX(MATRIX,S8,S9)
10	8	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	9	=INDEX(MATRIX,S9,S10)
11	9	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	10	=INDEX(MATRIX,S10,S11)
12	10	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	11	=INDEX(MATRIX,S11,S12)
13	11	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	12	=INDEX(MATRIX,S12,S13)
14	12	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	13	=INDEX(MATRIX,S13,S14)
15	13	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	0	14	=INDEX(MATRIX,S14,S15)
16	14	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	15	=INDEX(MATRIX,S15,S16)
17	15	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	16	=INDEX(MATRIX,S16,S17)
18	16	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	=INDEX(MATRIX,S17,S18)
19																		Total	=SUM(T3:T18)
20																		Test	=S18

Table 3. Model Solution Visiting the Towns Puzzle

1\A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
2		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	Path
3	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	12	1
4	2	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	16	1
5	3	0	0	0	0	0	0	1	0	0	0	1	1	0	1	0	0	11	1
6	4	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	3	1
7	5	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	7	1
8	6	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	13	1
9	7	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	2	1
10	8	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	10	1
11	9	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	6	1
12	10	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	15	1
13	11	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	4	1
14	12	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	8	1
15	13	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	0	14	1
16	14	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	5	1
17	15	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	9	1
18	16	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	1
19																		Total	16
20																		Test	1

Figure 3. Visiting the Towns Solution



Discussion and Summary

Puzzles utilizing mathematics enjoy immense popularity. The phenomenon Sudoku is perhaps the most recent example, see for example (Friesen, Patterson, Harmel, 2010). H. E. Dudeney is one of best-known mathematical riddle developers of all time. In 1917 he published *Amusements in Mathematics*, which is a collection of 430 mathematical and logic puzzles. One of the puzzles titled "Visiting the Towns" present a challenging combinatorial problem. In this paper, we presented an optimization model approach to solving this puzzle. Kilpelainen (2012) noted that researchers do not apply computing power to puzzles to take the fun out of them; rather, a possible purpose is to "develop instructive methods" for general classes of problems. This is certainly true here.

Finally, Dudeney (1917) makes two interesting remarks in his puzzle description. First he teases "How many different routes are there from which he can select?" TSP problems do not restrict the roads between towns; a TSP problem composed of 16 towns would have (16-1)!/2 which is approximately equal to 6.538E+11 (Malkevitch, 2005). Indeed, TSPs are known to be NP-complete (Press, Flannery, Teukolsky, & Vetterling, 1988). Of course, with the roads restricted, the number of possible tours is much reduced.

Dudeney's second interesting, possibly cryptic, remark likely speaks to the puzzle's complexity: "This is an absurdly easy puzzle, if you go the right way to work" (Dudeney 1917). Logically, if a town has only two roads, one must be incoming and one must be exiting. This directly removes the choice of road from towns 1, 2, 5, 6, 7, 8 and indirectly removes any choice of road from towns 9, 10, 13, and 14. Starting from town 1 and proceeding *right* to town 12, there is a choice. Taking the *right-most* path, we arrive at town 16. From town 16, there are no further choices. Armed with this solution algorithm, the exact solution is easily verified. This is a good quality for recreational computing exercises to possess.

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